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Iterative computation of time-optimal control functions on the differential analyzer

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PREFACE

This work was prepared by Robert Lloyd Berg as a research project in partial fulfillment of the requirements for the Professional Degree, Aeronautical and Astronautical Engineer, at the University of Michigan, Ann Arbor, Michigan, August 1962.

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The author wishes to acknowledge his appreciation for the guidance of Professor Elmer G. Gilbert in the preparation of this research study.

ABSTRACT

The time-optimal control functions for sample control systems were computed using the iterative computational procedure proposed by L. W. Neustadt as adapted for differential analyzer solution. The computational procedure investigated was developed from the method of steepest ascent. Adaptation for differential analyzer solution required that finite sized steps be taken during the ascent vice the infinitely small sized steps permissible in theory. This presented one of the problems of the computational procedure. No optimization of the step size was attempted in this work, however, several criteria for the selection of the step size were used with success on the two-dimensional systems.

The optimal control functions for two-dimensional systems were readily computed using Neustadt's iterative computational procedure. Analytic work was performed to verify some computer solutions. Agreement of computer and analytic solutions was favorable. The optimal control function for the three-dimensional system investigated was not obtained. After sixty iterations there was no apparent convergence to an optimal control function. Some refinement of the computational method used in this work would be required to extend the problem solutions to higher order systems. No work was performed to determine the effect computer errors would have on problem solutions.

I. INTRODUCTION

There has been an abundance of literature written concerning the classical control problem, i. e., given a control system, assumed to be normal, such that

(I-1) $\dot{\vec{x}} = A(t)\,\vec{x} + B(t)\,\vec{u}$

where: $\vec{x}$ represents an n-dimension state vector

$A(t)$ represents an n x n matrix, not necessarily constant

$B(t)$ represents an n x r matrix, not necessarily constant, and

$\vec{u}$ represents the r-dimension control vector

what is the optimal control, $\vec{u}^{o}$, which will drive any arbitrary state of the system, $\vec{x}(t_0)$, to some specified final state, $\vec{x}(t_1)$, in the minimum time. It is well known that the desired control law is bang-bang and is of the form

(I-2) $\vec{u} = \operatorname{sgn}\,(B^T(t)\,X^T(t_0,t)\,\vec{\eta})$

where: $\vec{\eta}$ represents an n-dimension constant vector

$X(t_0,t)$ represents an n x n matrix satisfying:

$\dot{X} = A(t)\,X$, where $X(t_0,t_0) = I$

Assuming $\| u_i \| \leqslant 1$, where u_i are the components of $\vec{u}$, then let $\vec{\alpha} = B^T(t)\,X^T(t_0,t)\,\vec{\eta}$, then $(\operatorname{sgn}\vec{\alpha})_i = +1$, $\alpha_i > 0$, $(\operatorname{sgn}\alpha)_i = -1$, $\alpha_i < 0$, and $(\operatorname{sgn}\alpha)_i$ is undefined for $\alpha_i = 0$.

Lucien Neustadt published a method for the synthesis of the control function which was adaptable to analog computer and digital computer solution, refer to L. W. Neustadt, "Synthesizing Time Optimal Control Systems," <u>Journal of Math. Analysis and Applications</u>, vol. 1, pp. 484-493, 1963. For the research

study reported in this paper, the proposed method was adapted for analog computer solution, and the regulator problem, $\vec{x}(t_1) = 0$, for three 2-dimension, time invariant systems, a neutrally stable system, a stable system, and a lightly damped system, as well as one 3-dimension, time invariant system were investigated with some analytic work developed to aid in the problem analysis.

The computational procedure for the above mentioned systems as adapted for analog computer solution can be briefly outlined as follows: given a control system as described by equations (I-1) and (I-2), define

(I-3) $$\vec{\zeta} = X^T(t_0,t)\,\vec{\eta}$$

then

(I-4) $$\dot{\vec{\zeta}} = -A^T(t)\vec{\zeta}, \quad \vec{\zeta}(0) = \vec{\eta}$$

Take some initial guess for $\vec{\eta}$ such that $\vec{\eta} \cdot \vec{x}(0) \leqslant 0$, solve this set of equations to obtain $\vec{\zeta}(t)$ from which, using equation (I-2)

(I-5) $$\vec{u} = \operatorname{sgn}\,(B^T(t)\ \vec{\zeta}(t))$$

Now form the function

(I-6) $$f(t_1,\vec{\eta}) = \vec{\eta} \cdot (\vec{z}(t_1,\vec{\eta}) + \vec{x}(0))$$

where by definition

(I-7) $$\vec{z}(t_1,\vec{\eta}) = \int_0^{t_1} X(t_0,s)\ B(s)\ \operatorname{sgn}\,(B^T(s)\ X^T(t_0,s)\vec{\eta})\ ds$$

so that

(I-8) $$f(t_1,\vec{\eta}) = \int_0^{t_1} \left\| \vec{\eta}^{\,T}\ X(t_0,s)\ B(s) \right\| ds + \vec{\eta} \cdot \vec{x}(0)$$

Determine the stopping time, t_1, from the zero crossing of $f(t_1,\vec{\eta})$ for $t = t_1$ when $f(t_1,\vec{\eta}) = 0$. Now, $0 \leqslant t_1 \leqslant t_1^{\circ}$, where the time t_1° is the minimum time to drive the system to its null state. Obtain the next trial value of $\vec{\eta}$ from the correction vector

(I-9) $$\Delta\vec{\eta} = (\vec{z}(t_1,\vec{\eta}) + \vec{x}(0))\Delta\tau$$

where $\vec{z}(t_1,\vec{\eta})$ is obtained by solving

(I-10) $$\dot{\vec{y}} = A(t)\ \vec{y} + B(t)\ \vec{u}\ ,\ \vec{y}(0) = 0$$

up to the time t_1, and then solving backward in time

(I-11) $\dot{\vec{w}} = A(t)\,\vec{w}\ ,\ \vec{w}(t_1) = \vec{y}(t_1)$

which yields

(I-12) $\vec{w}(0) = \vec{z}(t_1)$

As long as there is a correction vector, a new trial vector $\vec{\eta}$ may be used to make t_1 approach $t_1{}^o$ and $\vec{\eta}$ to approach $\vec{\eta}^{\,o}$ where $\vec{\eta}^{\,o}$ is the constant vector which yields the optimum control vector $\vec{u}^o$.

The outlined procedure can be adapted to the more general problem, however, the computer capacity required would necessarily be considerably extended. For this investigation of the general suitability of the computational method involved, it was decided to first investigate only the more simple types of problems already mentioned.

II. STATEMENT OF THE PROBLEM

Given a control system, assumed to be normal, with constant coefficients and a single input, 1) determine the optimum control law on the differential analyzer using an iterative technique proposed by Lucien Neustadt, 2) investigate the influence of the initial choice of the constant vector $\vec{\eta}$, 3) investigate the effect of the size of the sampling interval $\Delta\tau$, and 4) determine whether or not, for finite values of $\Delta\tau$, the constant vector $\vec{\eta}$ might leave the domain of $\vec{\eta}$, i. e., whether or not $\vec{\eta} \cdot \vec{x}(0) > 0$ for any iterative value of $\vec{\eta}$.

III. DISCUSSION

The solution to the classical regulator problem as researched for this paper follows with minor exceptions the method presented by Lucien Neustadt (Journal of Mathematical Analysis and Applications, vol. 1, pp 484-493). Given a control system such that

(III-1) $$\dot{\vec{x}} = A(t)\,\vec{x}(t) + B(t)\,\vec{u}(t)$$

where: $A(t)$ represents an n x n matrix, continuous function of time

$B(t)$ represents an n x r matrix, continuous function of time

$\vec{x}(t)$ represents the n-dimension state vector

$\vec{u}(t)$ represents the r-dimension control vector

what is the optimum control, $\vec{u}^{o}(t)$, which will drive the system from any arbitrary state, $\vec{x}(t_0)$, at time t_0 to some desired final state, $\vec{x}(t_1)$, at time t_1 in the minimum time. It is assumed that the admissible control is magnitude limited, $|u_i| \leq 1$, and that the system is a normal system. A system is normal if any component of $B^T(t)\,X^T(t_0,t)\vec{\eta} = 0$ on any finite interval of time, $0 \leq t \leq \infty$, implies $\vec{\eta} = 0$, where $\vec{\eta}$ is an n-dimension constant vector and $X(t,t_0)$ is the n x n transition matrix for

(III-2) $$\dot{\vec{x}} = A(t)\,\vec{x}(t)$$

satisfying the equations

(III-3) $$\dot{X}(t,t_0) = A(t)\,X(t,t_0)$$

(III-4) $$X(t_0,t_0) = I$$

For such a normal system the optimal control is unique and is bang-bang with $u_i(t) = \pm 1$, where the sign is given by

(III-5) $\quad \vec{u}(t) = \operatorname{sgn}\,(B^T(t)\; X^T(t_0,t)\,\vec{\eta})$

The solution for the system of equations described in (III-1) is given by

(III-6) $\quad \vec{x}(t) = X(t,t_0)\,\vec{x}(t_0) + X(t,t_0) \int_{t_0}^{t} X(t_0,s)\; B(s)\,\vec{u}(s)\; ds$

so that for $\vec{x}(t) = 0$

(III-7) $\quad -\vec{x}(t_0) = \int_{t_0}^{t} X(t_0,s)\; B(s)\,\vec{u}(s)\; ds$

Define the subset C(t):

(III-8) $\quad C(t) = \left\{ \int_{t_0}^{t} X(t_0,s)\; B(s)\,\vec{u}(s)\; ds \;,\; \vec{u}(s) \text{ admissible} \right\}$

where C(t) has the properties:

1) compact and convex

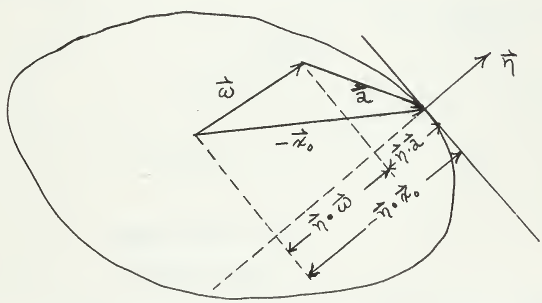
2) $C(t) \supset C(t_1)$ whenever $t > t_1$

3) C(t) grows continuously with time

For $t = t_1^{\,o}$, $-\vec{x}(0) \in C(t_1^{\,o})$ and is a boundary point of $C(t_1^{\,o})$. Since $C(t_1^{\,o})$ is a convex set there is at least one row vector, $\vec{\eta}^{\,T}$, such that

(III-9) $\quad -\vec{\eta}\cdot\vec{x}(0) \geqslant \vec{\eta}\cdot\vec{\omega}$, for all $\vec{\omega} \in C(t_1^{\,o})$

Thus $\vec{\eta}\cdot\vec{\omega}$ takes on its maximum value when $\vec{\omega} = -\vec{x}(0)$, see Figure III-1 for a two-dimension example.



Compact Set $C(t_1^{\,o})$

Figure III-1

Now $\vec{\omega} \in C(t_1{}^{o})$, so that

$$\text{(III-10)} \qquad \vec{\omega} = \int_{t_0}^{t_1{}^{o}} X(t_0,s)\, B(s)\, \vec{u}(s)\, ds\,, \quad |u_i| \leq 1$$

and

$$\text{(III-11)} \qquad \vec{\eta} \cdot \vec{\omega} = \int_{t_0}^{t_1{}^{o}} \vec{\eta}^{\,T} X(t_0,s)\, B(s)\, \vec{u}(s)\, ds\,, \quad |u_i| \leq 1$$

so to maximize $\vec{\eta} \cdot \vec{\omega}$, so that $\vec{\eta} \cdot \vec{\omega} = -\vec{\eta} \cdot \vec{x}(0)$, let

$$\text{(III-12)} \qquad u_i = \operatorname{sgn}\,(B^T(t)\, X^T(t_0,t)\, \vec{\eta})_i$$

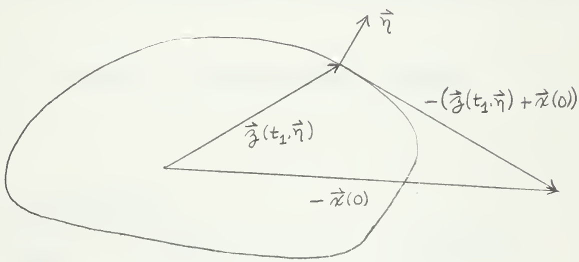
Now define

$$\text{(III-13)} \qquad \vec{z}(t_1,\vec{\eta}) = \int_{t_0}^{t_1} X(t_0,s)\, B(s)\, \operatorname{sgn}\,(B^T(s)\, X^T(t_0,s)\, \vec{\eta})\, ds$$

so that $\vec{z}(t_1,\vec{\eta})$ is on the boundary of $C(t_1)$, see Figure III-2. According to equation (III-9)

$$\text{(III-14)} \qquad \vec{\eta} \cdot \vec{z}(t_1,\vec{\eta}) > \vec{\eta} \cdot \vec{\zeta}\,, \text{ for all } \vec{\zeta} \in C(t_1),$$

$\vec{\zeta} \neq \vec{z}(t_1,\vec{\eta})$ and the function $\vec{\eta} \cdot \vec{z}(t_1,\vec{\eta})$ is a non-negative, non-decreasing monotonic function of time.



Compact Set $C(t_1)$

Figure III-2

Assuming the origin can be reached in some time, $t_1{}^{o}$, so that $-\vec{x}(0)$ lies on the boundary of the set $C(t_1{}^{o})$, there is a convex set of vectors, H_0, such that if $\vec{\eta} \in H_0$, $-\vec{\eta} \cdot \vec{x}(0)$ maximizes the function $\vec{\eta} \cdot \vec{\omega}$ for $\vec{\omega} \in C(t_1{}^{o})$ and $-\vec{x}(0) = \vec{z}(t_1{}^{o},\vec{\eta})$. Consider the function

(III-15) $\quad f(t, \vec{\eta}\ ;\vec{x}(0)) = \vec{\eta} \cdot (\vec{z}(t,\vec{\eta}) + \vec{x}(0))$

Now by definition, $\vec{z}(t_0,\vec{\eta}) = 0$, so restrict $\vec{\eta}$ such that

(III-16) $\quad f(t_0,\vec{\eta}\ ;\vec{x}(0)) \leqslant 0$

If $\vec{\eta} \notin H_0$, $f(t_1{}^{\circ},\vec{\eta}\ ;\vec{x}(0)) > 0$, hence for some time t_1, $0 \leqslant t_1 \leqslant t_1{}^{\circ}$,

(III-17) $\quad f(t_1,\vec{\eta}\ ;\vec{x}(0)) = 0$

Let

(III-18) $\quad F(\vec{\eta},\vec{x}(0)) \equiv t_1$

so that

(III-19) $\quad f(F(\vec{\eta},\vec{x}(0)),\vec{\eta}\ ;\vec{x}(0)) = 0$

Since $f(F(\vec{\eta},\vec{x}(0)),\vec{\eta}\ ;\vec{x}(0))$ is continuous in its arguments, $F(\vec{\eta},\vec{x}(0))$ must be continuous in $\vec{\eta}$ so that if $\vec{\eta} \notin H_0$, $F(\vec{\eta},\vec{x}(0)) \leqslant t_1{}^{\circ}$ and if $\vec{\eta} \in H_0$, $F(\vec{\eta},\vec{x}(0)) = t_1{}^{\circ}$. Thus any vector $\vec{\eta}$ such that $\vec{\eta} \cdot \vec{x}(0) \leqslant 0$, which maximizes the time t_1, for which $\vec{\eta} \cdot (\vec{z}(t_1,\vec{\eta}) + \vec{x}(0)) = 0$ where $\vec{z}(t_1,\vec{\eta})$ is given by equation (III-13) may be used in the optimum control function. Conversely, if $\vec{\eta}$ defines the optimal control function it maximizes the time.

To solve for $\vec{\eta}$ use the method of steepest ascent. Assume

(III-20) $\quad \vec{\eta} = \vec{\eta}(\tau)$

then

(III-21) $\quad d\vec{\eta}/d\tau = k\ \nabla F(\vec{\eta},\vec{x}(0))$

where $k > 0$ is some constant or function of τ and $\vec{\eta}$. In order to obtain a valid solution:

1) $F(\vec{\eta},\vec{x}(0))$ must have a maximum for $\vec{\eta}^{\circ}$

2) the solution $\vec{\eta}(\tau)$ never leaves the domain of $F(\vec{\eta},\vec{x}(0))$, and

3) the partial derivatives $\partial F/\partial\eta_i$ exist, where η_i are the components of $\vec{\eta}$.

Proceeding formally, see Neustadt's work for statements of proof, from equation (III-19), given a specified initial state, $\vec{x}(t_0)$,

where $F(\vec{\eta}, \vec{x}(0)) \equiv t_1$ is given implicitly by (III-19). Then

$$\nabla F = -\left(\frac{\partial f}{\partial \vec{\eta}} \Big/ \frac{\partial f}{\partial t}\right) \tag{III-23}$$

but

$$\frac{\partial f}{\partial \vec{\eta}} = (\vec{z}(t_1, \vec{\eta}) + \vec{x}(0)) \tag{III-24}$$

and

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\left(\int_{t_0}^{t} \left\| B^T(s)\, X^T(t_0,s)\, \vec{\eta} \right\| ds\right) \tag{III-25}$$

so that

$$\nabla F = -(\vec{z}(t_1, \vec{\eta}) + \vec{x}(0)) / \left\| B^T(t)\, X^T(t_0,t)\, \vec{\eta} \right\| \tag{III-26}$$

From equations (III-18) and (III-20),

$$F = F(\vec{\eta}(\tau)) \tag{III-27}$$

hence

$$\frac{\partial F}{\partial \tau} = \frac{\partial F}{\partial \vec{\eta}} \frac{\partial \vec{\eta}}{\partial \tau} = \nabla F \frac{d\vec{\eta}}{d\tau} \tag{III-28}$$

let

$$\frac{d\vec{\eta}}{d\tau} = \left\| B^T(t)\, X^T(t_0,t)\, \vec{\eta} \right\| \nabla F \tag{III-29}$$

then

$$\frac{d\vec{\eta}}{d\tau} = -(\vec{z}(t_1, \vec{\eta}) + \vec{x}(0)) \tag{III-30}$$

so that finally

$$\vec{\eta}_2 = \vec{\eta}_1 - (\vec{z}(t_1, \vec{\eta}_1) + \vec{x}(0))\, d\tau, \quad \vec{\eta}_2 - \vec{\eta}_1 = d\vec{\eta} \tag{III-31}$$

To compute this corrected vector, $\vec{\eta}_2$, it is necessary to precompute $\vec{z}(t_1, \vec{\eta}_1)$ for which in turn it is necessary to precompute the time t_1. Thus it is seen that $\vec{z}(t_1, \vec{\eta}_1)$ is not an instantaneous function of time and a finite sampling interval, $\Delta\tau$, is required, hence equation (III-31) becomes

$$\vec{\eta}_2 = \vec{\eta}_1 - (\vec{z}(t_1, \vec{\eta}_1) + \vec{x}(0))\, \Delta\tau \tag{III-32}$$

The time t_1 was defined by equation (III-17) and $\vec{z}(t_1, \vec{\eta}_1)$ is given by equation (III-13) so that we have

(III-33) $\quad \vec{\eta}_1 \cdot \vec{z}(t_1, \vec{\eta}_1) = \int_{t_0}^{t_1} \vec{\eta}_1^{\,T} X(t_0,s)\, B(s)\, \mathrm{sgn}\, (B^T(s)\, X^T(t_0,s) \vec{\eta}_1)\, ds$

or using the steering command given by (III-12)

(III-34) $\quad \vec{\eta}_1 \cdot \vec{z}(t_1, \vec{\eta}_1) = \int_{t_0}^{t_1} \| B^T(s) \; X^T(t_0,s) \; \vec{\eta}_1 \| \; ds$

Now let

(III-35) $\quad \vec{\zeta}(t) = X^T(t_0,t)\vec{\eta}_1$

then

(III-36) $\quad \dot{\vec{\zeta}} = -A^T(t)\vec{\zeta}(t) \;, \; \vec{\zeta}(t_0) = \vec{\eta}_1$

Solving equation (III-36) then gives $\vec{\zeta}(t)$ from which, together with equations (III-15), (III-17) and (III-34), it becomes possible to compute the time t_1. See Figure III-3 for a block diagram description of the computing technique. As revealed in Figure III-3, the control signal given by equation (III-12) is easily developed at the same time.

Now with the time t_1 known it becomes possible to compute $\vec{z}(t_1, \vec{\eta}_1)$. Consider

(III-37) $\quad \dot{\vec{y}} = A(t)\, \vec{y}(t) + B(t)\, \vec{u}(t) \;, \; \vec{y}(0) = 0$

which has the solution

(III-38) $\quad \vec{y}(t) = X(t,t_0) \int_{t_0}^{t} X(t_0,s)\, B(s)\, \vec{u}(s)\, ds$

which combined with equations (III-12) and (III-13) yields

(III-39) $\quad \vec{y}(t_1) = X(t_1,t_0)\; \vec{z}(t_1, \vec{\eta}_1)$

or alternately

(III-40) $\quad \vec{z}(t_1, \vec{\eta}_1) = X^{-1}(t_1,t_0)\; \vec{y}(t_1)$

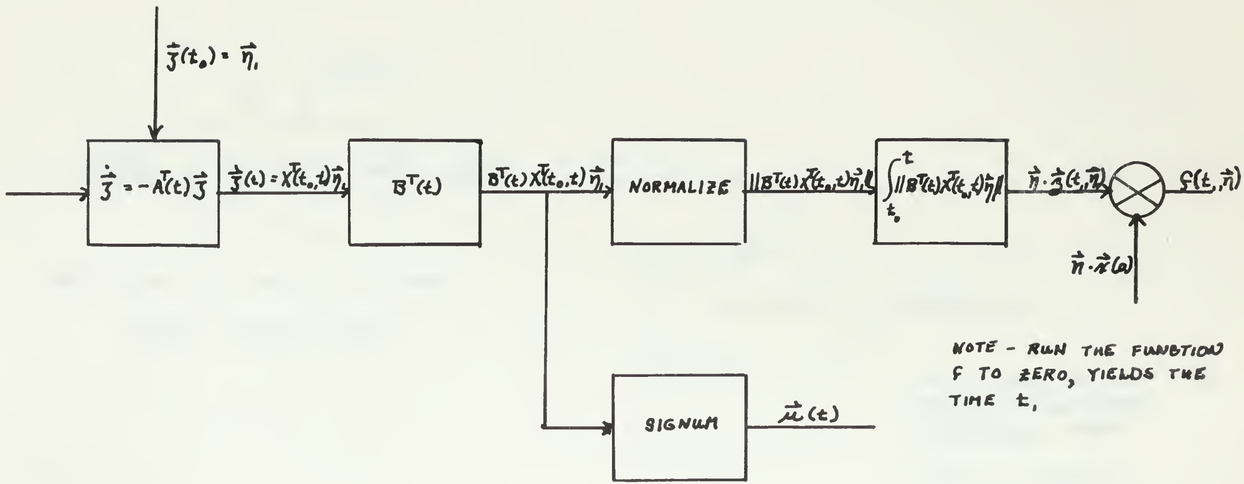
where $X^{-1}(t_1,t_0)$ is the transition matrix for

(III-41) $\quad \dot{\vec{w}} = A(t)\, \vec{w}(t) \;, \; \vec{w}(t_1) = \vec{y}(t_1)$

run backwards in time so that

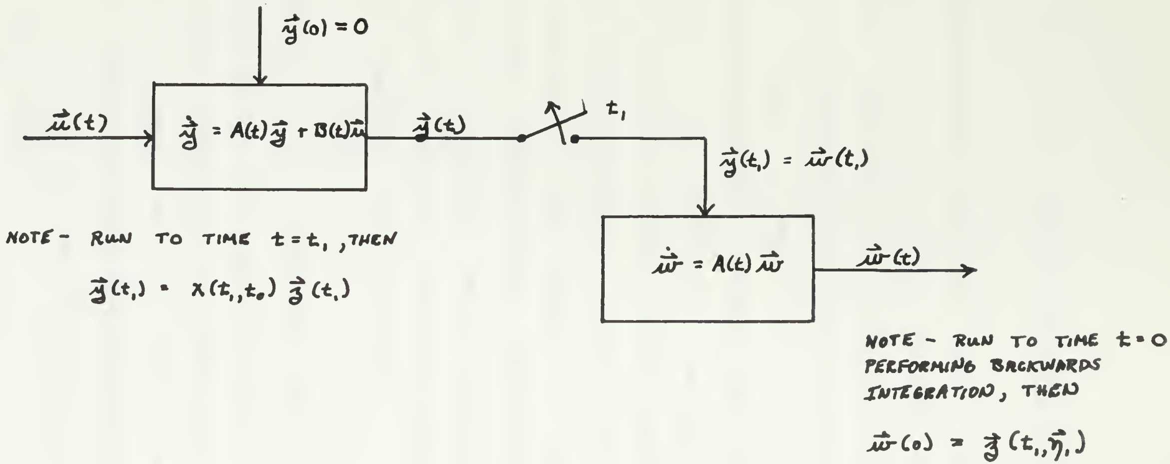
(III-42)
$$\begin{aligned} \vec{w}(t_0) &= X^{-1}(t_1,t_0)\; \vec{y}(t_1) \\ &= \vec{z}(t_1, \vec{\eta}_1) \end{aligned}$$

See Figure III-4 for a computational block diagram for the function $\vec{z}(t_1, \vec{\eta}_1)$.



BLOCK DIAGRAM FOR COMPUTATION OF THE TIME t_1

FIGURE III-3



BLOCK DIAGRAM FOR THE COMPUTATION OF $\vec{z}(t_1, \vec{\eta})$

FIGURE III-4

IV. COMPUTER DESCRIPTION AND CIRCUITRY

The analog computer used in the research of this paper was the Michigan twenty amplifier computer. The amplifiers had a gain of 100,000 to 1. Amplifiers were not drift stabilized, however, accurate balance was accomplished through use of a 0.1 volt full scale deflection voltmeter. Resistors were matched to provide an accuracy of 0.1% and capacitors were variable and adjusted to provide matched outputs. Twenty-three ten turn helipots plus a three gang helipot were set against a null helipot to provide accurate potentiometer settings. Individual problems were set up on removable patch boards.

In addition to the basic computer, a four place digital voltmeter was used to read out voltages. Automatic hold switching was accomplished by using relays in external circuitry.

The circuits presented in this section were developed for the general two-dimension, single input system:

(IV-1) $$\dot{\vec{x}} = A(t)\,\vec{x}(t) + \vec{b}(t)\,u(t) \quad , \quad |u| \leqslant 1$$

The three-dimension, single input system circuitry requires little modification and will not be presented in this section. For multiple input systems, a normalization scheme other than the one used in this work would be required. The specific circuitry for each problem analyzed is presented under the appropriate sub-section in Section V.

Given the system described by equation (IV-1), the adjoint set of equations is given by

(IV-2) $$\dot{\vec{\zeta}} = -A^T(t)\vec{\zeta}\,, \quad \vec{\zeta}(0) = \vec{\eta}_i$$

where

(IV-3) $$A^T(t) = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

The circuits required for the solution of this set of equations are simple integration circuits as presented in Figure IV-1, where

$$P_1 = R_1 C_{F_1} \frac{|a_{11}|}{\alpha_t}$$

$$P_2 = R_2 C_{F_1} \frac{|a_{21}| \quad \alpha_{\zeta_2}}{\alpha_{\zeta_1} \quad \alpha_t}$$

$$P_3 = R_3 C_{F_2} \frac{|a_{12}| \quad \alpha_{\zeta_1}}{\alpha_{\zeta_2} \quad \alpha_t}$$

$$P_4 = R_4 C_{F_2} \frac{|a_{22}|}{\alpha_t}$$

and

P_i = potentiometer setting

R_i = input resistance, ohms x 10^6

C_{F_i} = capacitance, farads x 10^{-6}

$$\alpha_e = \frac{e_{\text{physical quantity}}}{e_{\text{voltage representation}}}$$

$$\alpha_t = \frac{t_{\text{real time}}}{t_{\text{computer time}}}$$

now,

(IV-4) $$B^T(t)\, X^T(t_0,t)\, \vec{\eta} = B^T(t)\, \vec{\zeta}$$

or for the single input system

(IV-5) $$B^T(t)\, X^T(t_0,t)\, \vec{\eta} = \vec{b}^T(t)\, \vec{\zeta}$$

a simple summing circuit as shown in Figure IV-2 is required for the solution of the set of equations given by equation (IV-5), where

$$P_5 = \frac{R_5}{R_{F_1}} \frac{|b_1| \quad \alpha_{\zeta_1}}{\alpha_{B^T(t)\, X^T(t_0,t)\vec{\eta}}}$$

$$P_6 = \frac{R_6}{R_{F_1}} \frac{|b_2| \quad \alpha_{\zeta_2}}{\alpha_{B^T(t)\, X^T(t_0,t)\vec{\eta}}}$$

and

R_{F_i} = feedback resistance, ohms x 10^6

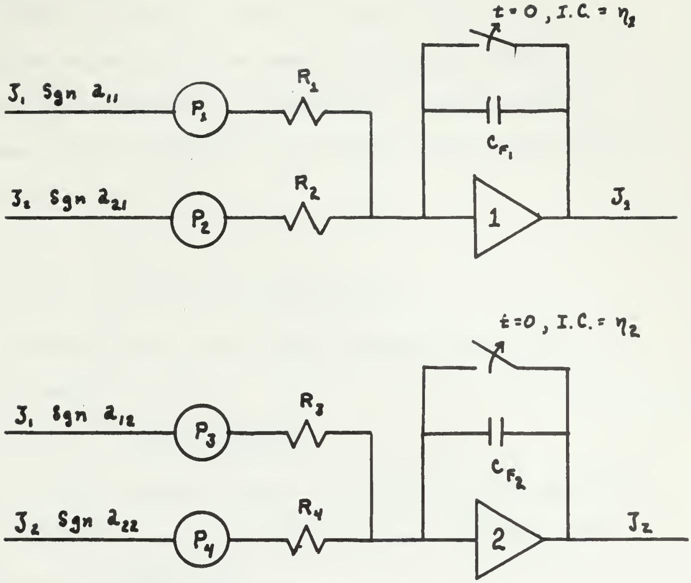


FIGURE III-1 CIRCUITS FOR ADJOINT SET OF EQUATIONS

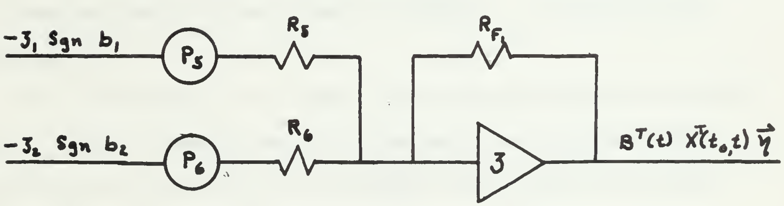


FIGURE IV-2 CIRCUIT FOR SOLUTION OF $B^T(t)\ X^T(t_0,t)\ \vec{\eta} = \vec{b}^T\vec{J}$

For the single input system

(IV-6) $$\left\| B^T(t)\; X^T(t_0,t)\, \vec{\eta} \right\| = \left| B^T(t)\; X^T(t_0,t)\, \vec{\eta} \right|$$

hence the absolute value circuit presented in Figure IV-3 was used to normalize the scalar quantity. To normalize a vector quantity a different scheme would be required. The absolute value circuit is an extremely accurate circuit.

Now then

(IV-7) $$\vec{\eta}\cdot\vec{z} = \int_{t_0}^{t_1} \left\| B^T(s)\; X^T(t_0,t)\, \vec{\eta} \right\| ds$$

Solution of this equation requires a simple integrating amplifier, see Figure IV-4, where

$$P_7 = R_7 C_{F_3} \qquad \frac{\propto B^T(t)\; X^T(t_0,t)\vec{\eta}}{\propto \vec{\eta}\cdot\vec{z} \quad \propto t}$$

By putting the initial condition equal to $+\ \vec{\eta}\cdot\vec{x}(0)$ on the amplifier the function $f(\vec{\eta},\vec{z};\vec{x}(0))$ can be obtained directly. To obtain the stopping time, run the system until

$$f(\vec{\eta},\vec{z};\vec{x}(0)) = 0$$

at which time switch to hold. Automatic hold switching was obtained through use of a relay as depicted in Figure IV-5. To insure accurate switching, the voltages were read out to four decimal places, an adjustable input was supplied to the amplifier as shown in Figure IV-5 and stopping times which satisfied $f(\vec{\eta},\vec{z};\vec{x}(0)) = \pm 0.0005$ volts were accepted.

Time was generated on a simple integrating amplifier with constant input. When the computer was run in reverse time the input voltage was reversed in polarity and the automatic hold circuit was switched to the output of the time generating amplifier. Again the system was automatically switched to hold at time $t = 0$.

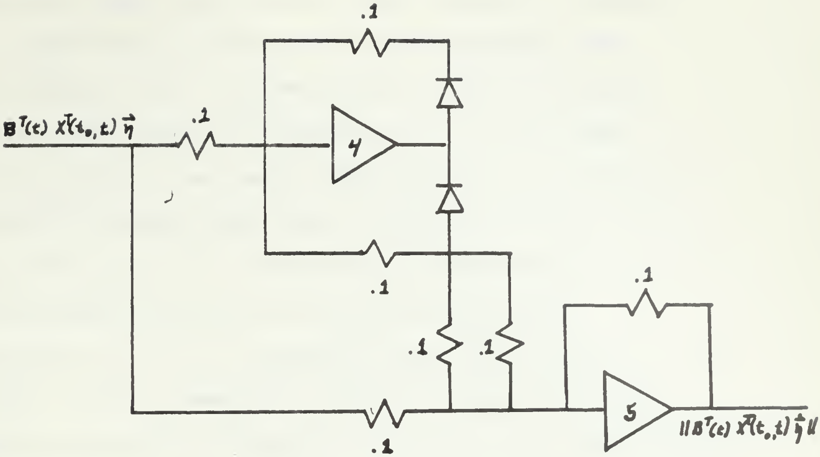


FIGURE IV-3 ABSOLUTE VALUE CIRCUIT

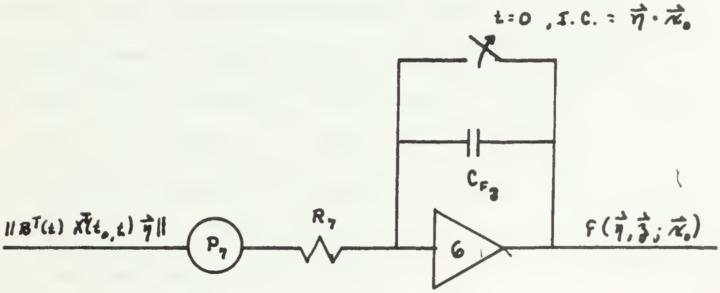


FIGURE IV-4 CIRCUIT FOR SOLUTION OF $f(\vec{\eta},\vec{\zeta};\vec{x}_0) = \int_0^t \|B^T(t)\,X^T(t_0,t)\,\vec{\eta}\|\,dt$

The control signal was produced by taking the output of amplifier number 3, Figure IV-2, and feeding that into the bang-bang circuit presented in Figure IV-6. The potentiometer is adjusted to produce the voltage representation of the magnitude of the control signal. The voltage potential of two common outputs will be that of the most positive of the individual amplifiers. At point A the voltage will be the most positive value of 1) + saturation, 2) - saturation or 3) u = -1 which automatically eliminates 2) as a possibility, since if $B^T(t)\ X^T(t_0,t)\vec{\eta} > 0$, $-1 >$ - saturation and if $B^T(t)\ X^T(t_0,t)\vec{\eta} < 0$, + saturation $> -1 >$ - saturation. Similarly at point B the voltage represents $u = +1$ for $B^T(t)\ X^T(t_0,t)\vec{\eta} > 0$, or $u = -1$ for $B^T(t)\ X^T(t_0,t)\vec{\eta} < 0$. This is an extremely accurate circuit for which the voltage representation for u was set accurately to four significant figures. The switching time was determined to be less than 0.1 milliseconds for an input changing at the rate of 50 volts/sec which was chosen as a representative rate of change of the voltage representation of $B^T(t)\ X^T(t_0,t)\vec{\eta}$.

The circuits used to compute $\vec{z}(t_1,\vec{\eta})$ were simple integrator circuits. First $\vec{y}(t_1)$ was computed, see Figure IV-7, from

(IV-8) $\quad \dot{\vec{y}} = A(t)\ \vec{y} + \vec{b}(t)\ u \quad , \quad \vec{y}(0) = 0$

and then $\vec{z}(t_1,\vec{\eta})$ was computed by integrating

(IV-9) $\quad \dot{\vec{w}} = A(t)\ \vec{w} \quad , \quad \vec{w}(t_1) = \vec{y}(t_1)$

backward in time, see Figure IV-8, to yield

(IV-10) $\quad \vec{z}(t_1,\vec{\eta}) = \vec{w}(0)$

To perform the backward integration, it was only necessary to reverse the polarity of all the inputs to the integrators used in the generation of $\vec{y}(t_1)$ and remove the control signal input, all of which was accomplished by manual switching while in the hold condition.

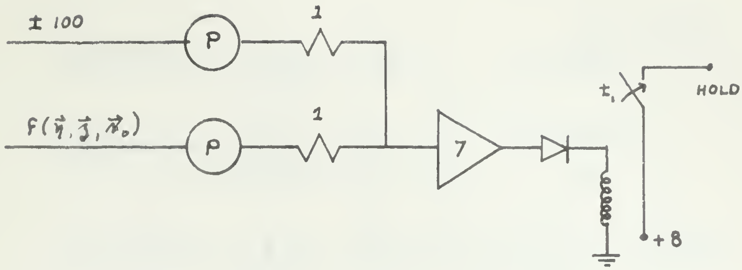


FIGURE IV-5 AUTOMATIC HOLD CIRCUIT

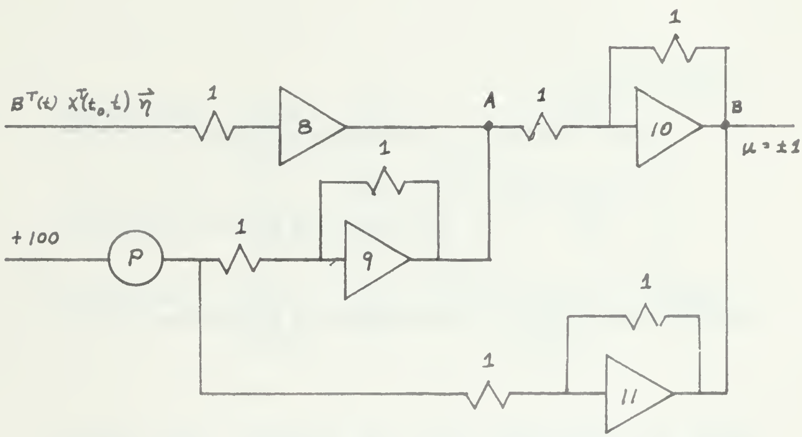


FIGURE IV-6 BANG-BANG CIRCUIT

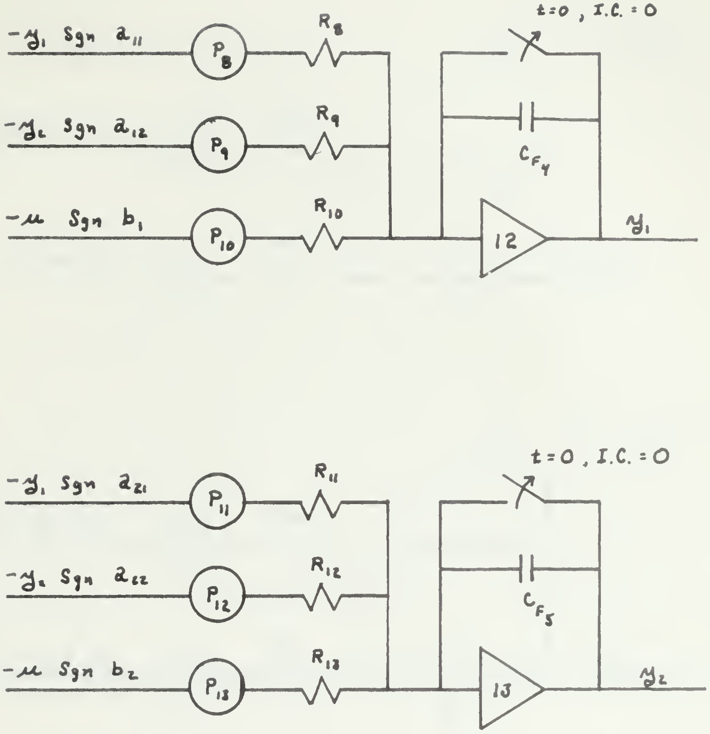


FIGURE IV-7 CIRCUITS FOR THE GENERATION OF $\vec{y}(t)$

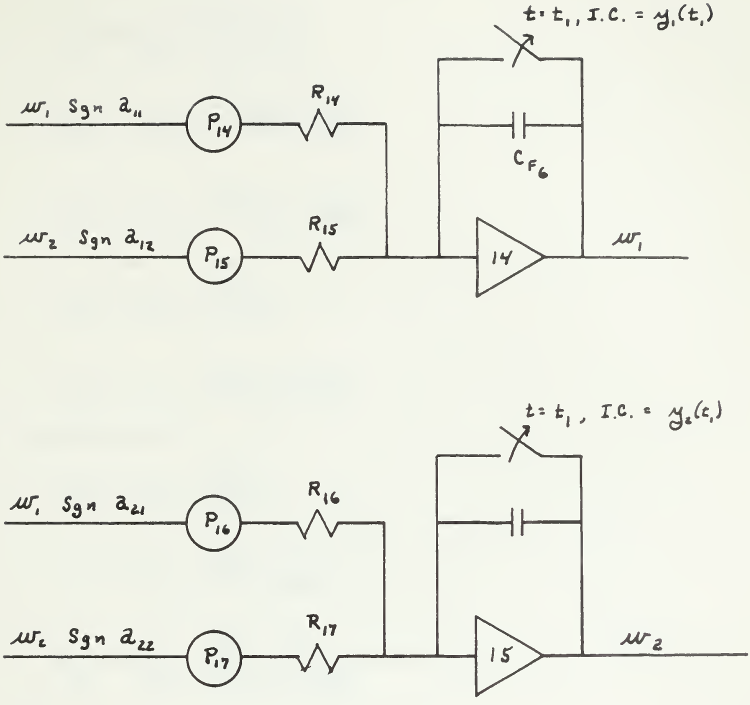


FIGURE IV - 8 CIRCUITS FOR THE GENERATION OF $\vec{w}(t)$

In Figure IV-7,

$$P_8 = R_8 C_{F_4} \frac{|a_{11}|}{\alpha_t}$$

$$P_9 = R_9 C_{F_4} \frac{|a_{12}| \ \alpha_{y_2}}{\alpha_{y_1} \ \alpha_t}$$

$$P_{10} = R_{10} C_{F_4} \frac{|b_1| \ \alpha_u}{\alpha_{y_1} \ \alpha_t}$$

$$P_{11} = R_{11} C_{F_5} \frac{|a_{21}| \ \alpha_{y_1}}{\alpha_{y_2} \ \alpha_t}$$

$$P_{12} = R_{12} C_{F_5} \frac{|a_{22}|}{\alpha_t}$$

$$P_{13} = R_{13} C_{F_5} \frac{|b_2| \ \alpha_u}{\alpha_{y_2} \ \alpha_t}$$

and in Figure IV-8,

$$P_{14} = R_{14} C_{F_6} \frac{|a_{11}|}{\alpha_t}$$

$$P_{15} = R_{15} C_{F_6} \frac{|a_{11}| \ \alpha_{w_2}}{\alpha_{w_1} \ \alpha_t}$$

$$P_{16} = R_{16} C_{F_7} \frac{|a_{21}| \ \alpha_{w_1}}{\alpha_{w_2} \ \alpha_t}$$

$$P_{17} = R_{17} C_{F_7} \frac{|a_{22}|}{\alpha_t}$$

V. SPECIFIC PROBLEM SOLUTIONS

Selected two-dimension as well as one three-dimension system were investigated for the research of this paper. The two-dimension systems investigated were:

1) a neutrally stable system, $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u \quad , \; |u| \leq 1$

2) a stable system, $\dot{\vec{x}} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u \quad , \; |u| \leq 1$

3) a lightly damped system , $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -.2 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u \quad , \; |u| \leq 1$

and the three-dimension system investigated was:

4) $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} u \quad , \; |u| \leq 1$

Plant 1) was chosen to provide a system on which a rather thorough analytic solution could be obtained in an effort to gain some insight into the problem solution. Plant 2) was chosen to investigate the effect of running a stable plant in reverse time, that is a stable and an unstable combination. Plant 3) was chosen as a representative two-dimension system for which solution might be desired. The three-dimension plant 4) was selected to provide a simple extension of the computational procedure to higher order systems.

Plant 1)

The analytic solution to the control problem synthesis parallels the analog computer solution. Given

(V-1) $$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u \quad , \; |u| \leq 1$$

define

(V-2) $$\dot{\vec{\zeta}} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \vec{\zeta} \; , \quad \vec{\zeta}(0) = \vec{\eta}_1 = \begin{Bmatrix} \eta_{11} \\ \eta_{12} \end{Bmatrix}$$

where

(V-3) $$\vec{\zeta} = X^T(t_0,t)\vec{\eta}_1 \quad , \; X(t_0,t_0) = I$$

then

(V-4) $\qquad B^T(t)\, X^T(t_0,t)\, \vec{\eta}_1 = \vec{b}^T \vec{\zeta} = -\eta_{11} t + \eta_{12}$

from which

(V-5) $\qquad u = \mathrm{sgn}\,(-\eta_{11} t + \eta_{12})$

and

(V-6) $\qquad \vec{\eta}_1 \cdot \vec{z} = \int_{t_0}^{t_1} |-\eta_{11} s + \eta_{12}|\; ds$

Referring to Figure V-1, for η_{11} and η_{12} of the same sign

(V-6 A) $\qquad \vec{\eta}_1 \cdot \vec{z} = |\eta_{12}|\; t - |\eta_{11}| \frac{t^2}{2}\,, \quad t \leqslant \eta_{12}/\eta_{11}$

(V-6 B) $\qquad \vec{\eta}_1 \cdot \vec{z} = -|\eta_{12}|\; t + |\eta_{11}| \frac{t^2}{2}\,, \quad t \leqslant \eta_{12}/\eta_{11}$

and for η_{11} and η_{12} of opposite signs

(V-6 C) $\qquad \vec{\eta}_1 \cdot \vec{z} = |\eta_{12}|\; t + |\eta_{11}| \frac{t^2}{2}\,, \quad t \geqslant 0$

Now

(V-7) $\qquad f(\vec{\eta}_1, \vec{z}; \vec{x}_0) = 0$

where $\vec{x}_0 = \vec{x}(0) = \begin{Bmatrix} x_{01} \\ x_{02} \end{Bmatrix}$, defines the time t_1 implicitly. From (V-6 A)

(V-7 A) $\qquad t_1 = \dfrac{|\eta_{12}| - (|\eta_{12}|^2 + 2|\eta_{11}|\,(\eta_{11}\, x_{01} + \eta_{12} x_{02}))^{\frac{1}{2}}}{|\eta_{11}|}$

from (V-6 B)

(V-7 B) $\qquad t_1 = \dfrac{|\eta_{12}| + (|\eta_{12}|^2 - 2|\eta_{11}|\,(\eta_{11} x_{01} + \eta_{12} x_{02} + |\eta_{12}|^2/|\eta_{11}|))^{\frac{1}{2}}}{|\eta_{11}|}$

and from (V-6 C)

(V-7 C) $\qquad t_1 = \dfrac{-|\eta_{12}| + (|\eta_{12}|^2 - 2|\eta_{11}|\,(\eta_{11} x_{01} + \eta_{12} x_{02} + |\eta_{12}|^2/|\eta_{11}|))^{\frac{1}{2}}}{|\eta_{11}|}$

By definition

(V-8) $\qquad \vec{z}(t_1, \vec{\eta}_1) = \int_{t_0}^{t_1} X(t_0,s)\, B(s)\, \mathrm{sgn}\,(B^T(s)\, X^T(t_0,s)\, \vec{\eta}_1)\; ds$

but

(V-9) $\qquad X(0,s) = \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix}$

so

(V-10) $\qquad \vec{z}(t_1, \vec{\eta}_1) = \int_0^{t_1} \begin{Bmatrix} -s \\ 1 \end{Bmatrix} \mathrm{sgn}\,(-\eta_{11}\, s + \eta_{12})\; ds$

so that for $t \leqslant \eta_{12}/\eta_{11}$, and η_{11} and η_{12} of the same sign

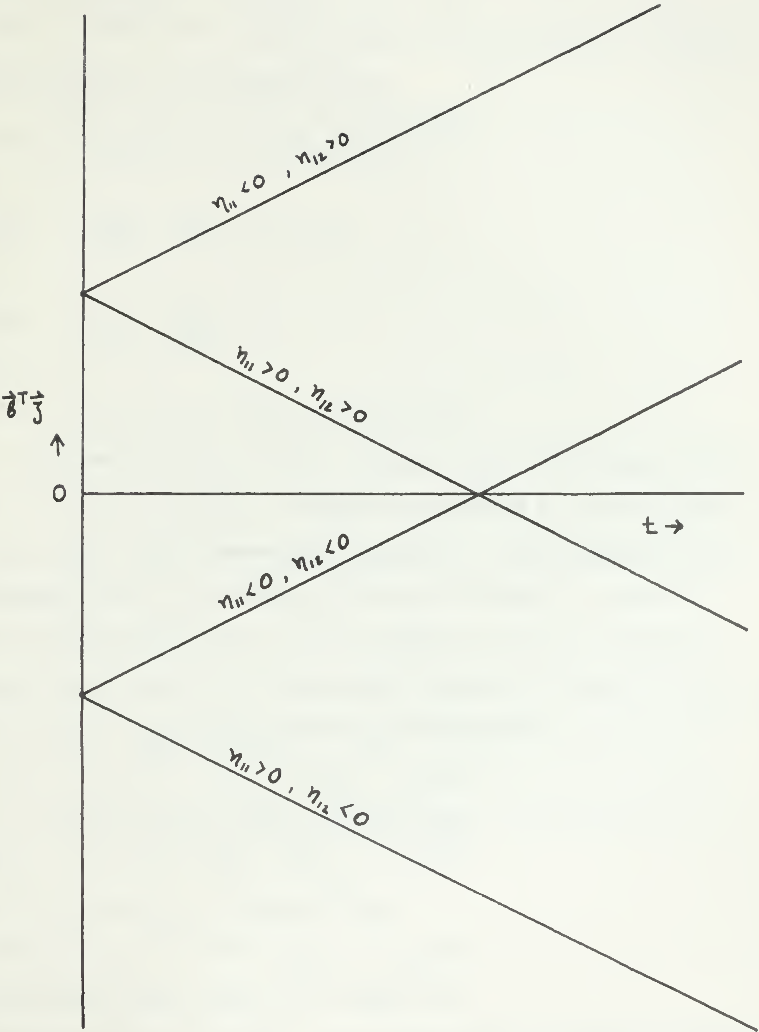


FIGURE V-1 PLOTS OF: $-\eta_{11} t + \eta_{12} = \vec{B}^T\vec{J}$

(V-10 A) $\qquad \vec{z}(t_1, \vec{\eta}_1) = \begin{Bmatrix} -(t_1^{\,2})/2 \ \mathrm{sgn}\ (\eta_{12}) \\ t_1 \ \mathrm{sgn}\ (\eta_{11}) \end{Bmatrix}$

for $t_1 \geqslant \eta_{12}/\eta_{11}$, and η_{11} and η_{12} of the same sign

(V-10 B) $\qquad \vec{z}(t_1, \vec{\eta}_1) = \begin{Bmatrix} -\frac{1}{2}(\eta_{12}/\eta_{11})^2 \ \mathrm{sgn}\ (\eta_{12}) - \frac{1}{2}(\eta_{12}/\eta_{11}) \ \mathrm{sgn}\ (\eta_{11}) + t_1^2/2 \ \mathrm{sgn}\ (\eta_{11}) \\ \eta_{12}/\eta_{11} \ \mathrm{sgn}\ (\eta_{12}) + \eta_{12}/\eta_{11} \ \mathrm{sgn}\ (\eta_{11}) - t_1 \ \mathrm{sgn}\ (\eta_{11}) \end{Bmatrix}$

and for η_{11} and η_{12} of opposite signs

(V-10 C) $\qquad \vec{z}(t_1, \vec{\eta}_1) = \begin{Bmatrix} (t_1^{\,2})/2 \ \mathrm{sgn}\ (\eta_{11}) \\ -t_1 \ \mathrm{sgn}\ (\eta_{11}) \end{Bmatrix}$

Now

(V-11) $\qquad \frac{d\eta_1}{d\tau} = -(\vec{z}(t_1, \vec{\eta}_1) + \vec{x}_0)$

and

(V-12) $\qquad \Delta\eta_1 = \frac{d\eta_1}{d\tau} \Delta\tau$

so that

(V-13) $\qquad \eta_{i+1} = \eta_i - (\vec{z}(t_1, \eta_1) + \vec{x}_0) \ \Delta\tau$

With this computational procedure, successive values of $\vec{\eta}$ were obtained for $\Delta\tau = 1$, $\vec{x}_0 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$. The data are presented in Table V-1 and a plot of η_{i2} versus η_{i1} is presented in Figure V-2. From equations (V-7) it is apparent that t_1 is a function of only one variable. To gain a better understanding of the function $F(\vec{\eta}, \vec{x}_0)$ on which the method of steepest ascent was used, it was decided to find $F(\rho)$ where $\rho = \eta_{12}/\eta_{11}$. Then for $t_1 \leqslant \eta_{12}/\eta_{11}$, η_{11} and η_{12} the same sign

(V-14 A) $\qquad t_1 = \rho - (\rho^2 - 2(x_{01} + \rho \, x_{02}))^{\frac{1}{2}}$

for $t_1 \geqslant \eta_{12}/\eta_{11}$, η_{11} and η_{12} the same sign

(V-14 B) $\qquad t_1 = \rho + (-\rho^2 + 2(x_{01} + \rho \, x_{02}))^{\frac{1}{2}}$

and for η_{11} and η_{12} of opposite signs

(V-14 C) $\qquad t_1 = \rho + (\rho^2 + 2(x_{01} + \rho \, x_{02}))^{\frac{1}{2}}$

$F(\vec{\eta}_1, \vec{x}_0)$ was plotted versus ρ in Figure V-3.

The effect of the size of $\Delta\tau$ was investigated by linearizing equation (V-13) about $\vec{\eta}^{\,o} = \begin{Bmatrix} -1 \\ -1 \end{Bmatrix}$ which was obtained from Figure V-3. Since η_{11}

TABLE V-I ANALYTIC DATA FOR $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u$

RUN	η_1	η_2	t_1	$J_1(t_1)$	$J_2(t_2)$	$d\eta_1/d\tau$	$d\eta_2/d\tau$	$\Delta\eta_1$	$\Delta\eta_2$
$\Delta\tau = 1.0$									
1	0	.5	0	0	0	-1.0	0	-1.0	0
2	-1.0	.5	1.0	-.5	+1.0	-.5	-1.0	-.5	-1.0
3.	-1.5	-.5	1.706	-1.341	+1.040	+.342	-1.040	+.342	-1.040
4	-1.158	-1.540	1.810	+.132	-.850	-1.132	+.850	-1.132	+.850
5	-2.290	-.690	1.592	-1.340	+1.089	+.340	-1.089	+.340	-1.089
6	-1.950	-1.779	1.992	-1.157	+.168	+.157	-.168	+.157	-.168
7	-1.793	-1.947	1.991	-.806	-.075	-.194	+.075	-.194	+.075
8	-1.987	-1.872	1.997	-1.105	+.111	+.105	-.111	+.105	-.111
9	-1.882	-1.983	1.995	-.888	-.111	-.112	+.111	-.112	+.111
10	-1.994	-1.872	1.996	-1.109	+.120	+.109	-.120	+.109	-.120
11	-1.885	-1.992	1.997	-.876	-.117	-.124	+.117	-.124	+.117
12	-2.009	-1.875	2.000	-1.129	+.132	+.129	-.132	+.129	-.132
$\Delta\tau = 0.375$									
1	-1.0	0	1.414	-1.0	+1.414	0	-1.414	0	-.531
2	-1.0	-.531	1.839	-1.406	+.777	+.406	-.777	+.152	-.292
3	-.848	-.823	1.999	-1.057	+.059	+.057	-.059	+.021	-.022
4	-.827	-.845	2.000	-.957	-.042	-.043	+.042	-.016	+.016
5	-.843	-.829	2.000	-1.034	+.014	+.034	-.014	+.013	-.005
6	-.830	-.834	2.000	-.990	-.008	-.010	+.008	-.004	+.003

TABLE V-I (CONTINUED)

RUN	η_1	η_2	t_1	$\mathcal{J}_1(t_1)$	$\mathcal{J}_2(t_1)$	$d\eta_1/d\tau$	$d\eta_2/d\tau$	$\Delta\eta_1$	$\Delta\eta_2$
$\Delta\tau = 0.25$									
1	-1.0	0	1.414	-1.0	+1.414	0	-1.414	0	-.353
2	-1.0	-.353	1.722	-1.357	+1.016	+.357	-1.016	+.089	-.254
3	-.911	-.607	1.921	-1.396	+.567	+.396	-.567	+.099	-.142
4	-.812	-.749	1.994	-1.138	+.148	+.138	-.148	+.034	-.037
5	-.778	-.786	1.999	-.975	-.023	-.025	+.023	-.006	+.006
$\Delta\tau = 0.1$									
1	DATA	FOR	$\Delta\tau$ =	1.0					
2	DATA	FOR	$\Delta\tau$ =	1.0					
3	-1.500	-.580	1.707	-1.346	+1.141	+.346	-1.141	+.035	-.114
4	-1.465	-.614	1.769	-1.386	+.931	+.386	-.931	+.039	-.093
5	-1.426	-.707	1.820	-1.409	+.828	+.409	-.828	+.041	-.083
6	-1.385	-.790	1.868	-1.426	+.728	+.426	-.728	+.043	-.073
7	-1.342	-.863	1.903	-1.401	+.619	+.401	-.619	+.040	-.062
8	-1.302	-.925	1.943	-1.386	+.523	+.386	-.523	+.039	-.052
9	-1.263	-.977	1.958	-1.320	+.408	+.320	-.408	+.032	-.041
10	-1.231	-1.018	1.973	-1.271	+.325	+.271	-.325	+.027	-.032
11	-1.204	-1.050	1.985	-1.209	+.241	+.209	-.241	+.021	-.024
12	-1.183	-1.074	1.992	-1.163	+.180	+.163	-.180	+.016	-.018

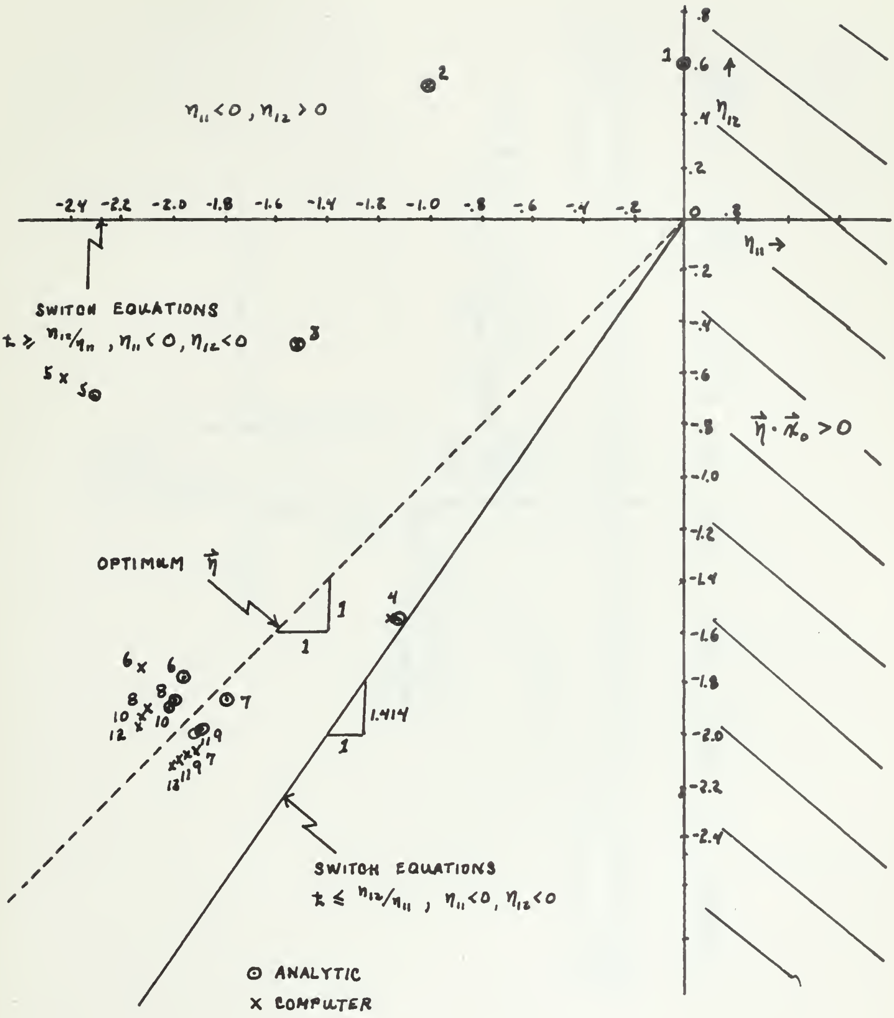


FIGURE V-2 PLOT OF η_{i2} VERSUS η_{i1}

FOR $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \mu$

$\vec{x}_0 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$, $\Delta\tau = 1.0$

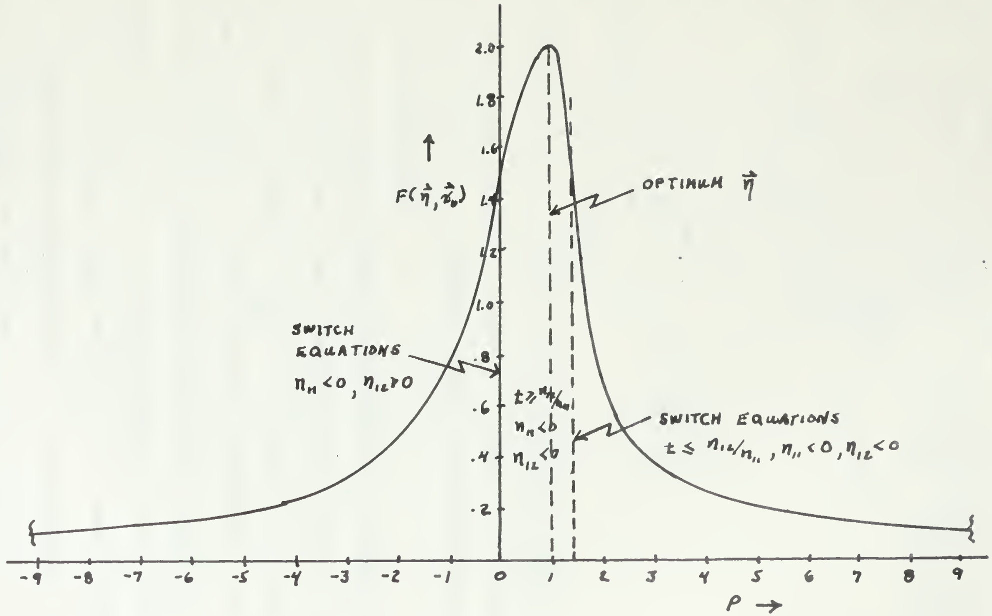


FIGURE V-3 $F(\vec{\eta}, \vec{x}_0)$ VERSUS ρ FOR $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u$, $\vec{x}_0 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$

and $\eta_{12} < 0$ and $t_1 > \eta_{12}/\eta_{11}$, using equations (V-10 B) and (V-13)

(V-13 B) $$\vec{\eta}_{i+1} = \begin{Bmatrix} -\left[(\eta_{12}/\eta_{i1})^2 - \eta_{i2}/\eta_{i1}\sqrt{-(\eta_{i2}/\eta_{i1})^2 + 2(x_{01} + \eta_{i2}/\eta_{i1}\, x_{02})} + x_{02}\right]\Delta\tau + \eta_{i1} \\ -\left[-(\eta_{12}/\eta_{i1}) + \sqrt{-(\eta_{i2}/\eta_{i1})^2 + 2(x_{01} + \eta_{i2}/\eta_{i1}\, x_{02})} + x_{02}\right]\Delta\tau + \eta_{i2} \end{Bmatrix}$$

define

(V-15) $$\vec{\delta} = \vec{\eta}_{i+1} - \vec{\eta}_i = \left.\frac{\partial \vec{\eta}_{i+1}}{\partial \vec{\eta}}\right|_{\vec{\eta} = \vec{\eta}^0} d\vec{\eta}$$

then for $\vec{x}_0 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$, $\vec{\eta}^0 = \begin{Bmatrix} -1 \\ -1 \end{Bmatrix}$

(V-16) $$\vec{\delta} = A\, d\vec{\eta}$$

where

(V-17) $$A = \begin{bmatrix} (1 - 2\Delta\tau) & 2\Delta\tau \\ 2\Delta\tau & (1 - 2\Delta\tau) \end{bmatrix}$$

The eigenvalues of A are

(V-18) $$\lambda = 1,\ 1 - 4\Delta\tau$$

For stability, $|\lambda_i| < 1$, hence for the given plant with $\vec{x}_0 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$ $\Delta\tau < 0.50$ gives stability. As noted in Table V-1 and Figure V-2 with $\Delta\tau = 1$ there is a slight divergence in the successive values of $\vec{\eta}$ as $\vec{\eta}^0$ was approached.

Using the circuitry presented in Section IV as combined in Figure V-4, the computational procedure was investigated for values of $\Delta\tau$ = 1.0, 0.375, 0.25 and 0.1 with $\vec{x}_0 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$. The data for the iterations are presented in Table V-II and are plotted in Figures V-2, V-5, V-6 and V-7 respectively with the corresponding analytically computed iterations. As expected $\Delta\tau = 0.25$, $\lambda = 1$ and 0, yielded the most positive convergence to $\vec{\eta}^0$, $\Delta\tau = 0.375$ gave rapid convergence with some oscillations, $\Delta\tau = 0.1$ gave slow convergence and $\Delta\tau = 1.0$ gave large oscillations. The hill climbing pattern associated with each value of $\Delta\tau$ is presented in Figure V-8. It was noted in Table V-II that as $\vec{x} \rightarrow 0$, the correction vector $(\vec{z}(t_1, \vec{\eta}_1) + \vec{x}_0)\Delta\tau \longrightarrow 0$ independent of the choice of $\Delta\tau$. The plots of $x_2(t)$ versus $x_1(t)$ are presented in Figures V-9, V-10, V-11 and

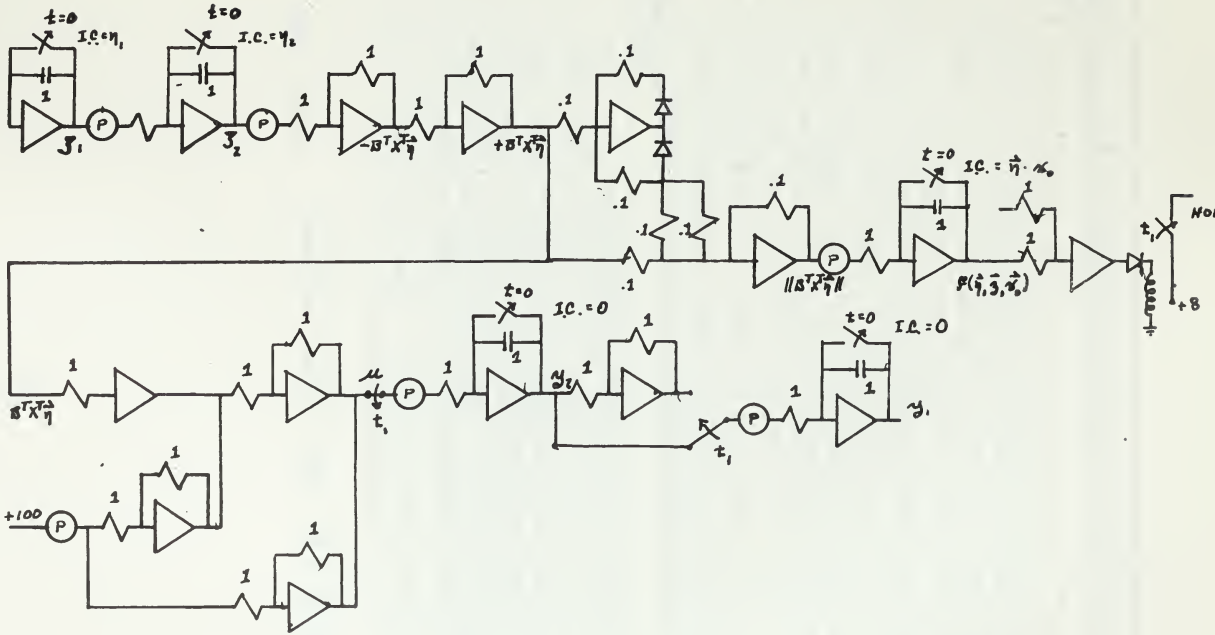


FIGURE V-4 SCHEMATIC DIAGRAM FOR $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \mu$

RUN	η_1	η_2	t_1	$x_1(t_1)$	$x_2(t_1)$	$z_1(t_1)$	$z_2(t_1)$	$d\eta_1/d\tau$	$d\eta_2/d\tau$	$\Delta\eta_1$	$\Delta\eta_2$	P
$\Delta\tau = 1.0$												
1	0	.5	0	1	0	0	0	-1.0	0	-1.0	0	∞
2	-1.0	.5	1.704	+1.506	+1.007	-.500	+1.007	-.500	-1.007	-.500	-1.007	-.500
3	-1.500	-.507	1.705	+1.449	+1.051	-1.351	+1.039	+.351	-1.039	+.351	-1.039	+.338
4	-1.139	-1.546	1.786	-.419	-.956	+.275	-.942	-1.275	+.942	-1.275	+.942	+1.357
5	-2.414	-.604	1.649	+1.581	+1.133	-1.289	+1.137	+.289	-1.137	+.289	-1.137	+.250
6	-2.125	-1.741	1.986	+.365	+.322	-1.216	+.326	+.216	-.326	+.216	-.326	+.819
7	-1.909	-2.063	2.000	-.175	-.180	-.805	-.178	-.195	+.178	-.195	+.178	+1.080
8	-2.104	-1.885	1.982	+.194	+.185	-1.165	+.188	+.165	-.188	+.165	-.188	+.896
9	-1.939	-2.073	1.989	-.151	-.164	-.828	-.153	-.172	+.153	-.172	+.153	+1.069
10	-2.111	-1.920	2.003	+.171	+.162	-1.139	+.162	+.139	-.162	+.139	-.162	+.910
11	-1.972	-2.082	2.009	-.126	-.125	-.861	-.125	-.139	+.125	-.139	+.125	+1.055
12	-2.111	-1.957	2.012	+.131	+.130	-1.115	+.129	+.115	-.129	+.115	-.129	+.926
$\Delta\tau = 0.375$												
1	-1.0	0	1.409	+1.993	+1.408	-.989	+1.420	-.011	-1.420	-.004	-.533	0
2	-1.004	-.533	1.841	+1.011	+.770	-1.391	+.770	+.391	-.770	+.147	-.288	+.531
3	-.857	-.821	1.993	+.075	+.071	-1.058	+.071	+.058	-.071	+.022	-.027	+.956
4	-.835	-.848	1.993	-.039	-.041	-.950	-.043	-.050	+.043	-.019	+.016	+1.017
5	-.854	-.832	1.999	+.045	+.043	-1.036	+.043	+.036	-.043	+.013	-.016	+.973
6	-.841	-.848	2.000	-.023	-.023	-.969	-.025	-.031	+.025	-.016	+.009	+1.009

TABLE V-II COMPUTER DATA FOR $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u$

TABLE V-II (CONTINUED)

RUN	η_1	η_2	t_1	$x_1(t_1)$	$x_2(t_1)$	$y_1(t_1)$	$y_2(t_1)$	$d\eta_1/d\tau$	$d\eta_2/d\tau$	$\Delta\eta_1$	$\Delta\eta_2$	ρ
$\Delta\tau = 0.25$												
1	-1.0	0	1.409	+1.993	+1.408	-.989	+1.420	-.011	-1.420	-.002	-.355	0
2	-1.002	-.355	1.721	+1.328	+1.005	-1.359	+1.005	+.359	-1.005	+.087	-.252	+.354
3	-.915	-.607	1.914	+.720	+.576	-1.385	+.578	+.385	-.578	+.091	-.145	+.674
4	-.824	-.752	1.995	+.169	+.138	-1.147	+.158	+.147	-.158	+.037	-.039	+.912
5	-.787	-.791	2.000	-.023	-.023	-.969	-.025	-.031	+.025	-.012	+.009	+1.009
$\Delta\tau = 0.1$												
1	DATA	FOR	$\Delta\tau = 1.0$									
2	DATA	FOR	$\Delta\tau = 1.0$									
3	-1.500	-.507	1.705	+1.449	+1.051	-1.351	+1.039	+.351	-1.039	+.035	-.104	+.338
4	-1.465	-.611	1.765	+1.258	+.933	-1.378	+.933	+.378	-.933	+.038	-.093	+.417
5	-1.427	-.704	1.829	+1.115	+.840	-1.407	+.832	+.407	-.832	+.041	-.083	+.493
6	-1.386	-.787	1.853	+.935	+.725	-1.406	+.725	+.406	-.725	+.041	-.073	+.568
7	-1.345	-.860	1.912	+.778	+.620	-1.406	+.620	+.406	-.620	+.041	-.062	+.640
8	-1.304	-.912	1.931	+.643	+.522	-1.378	+.522	+.378	-.522	+.038	-.052	+.699
9	-1.266	-.964	1.942	+.502	+.427	-1.330	+.427	+.330	-.427	+.033	-.043	+.760
10	-1.233	-1.007	1.969	+.382	+.337	-1.282	+.336	+.282	-.336	+.028	-.034	+.814
11	-1.205	-1.040	1.982	+.280	+.255	-1.247	+.257	+.247	-.257	+.025	-.026	+.862
12	-1.180	-1.066	1.989	+.186	+.179	-1.168	+.180	+.168	-.180	+.017	-.018	+.902

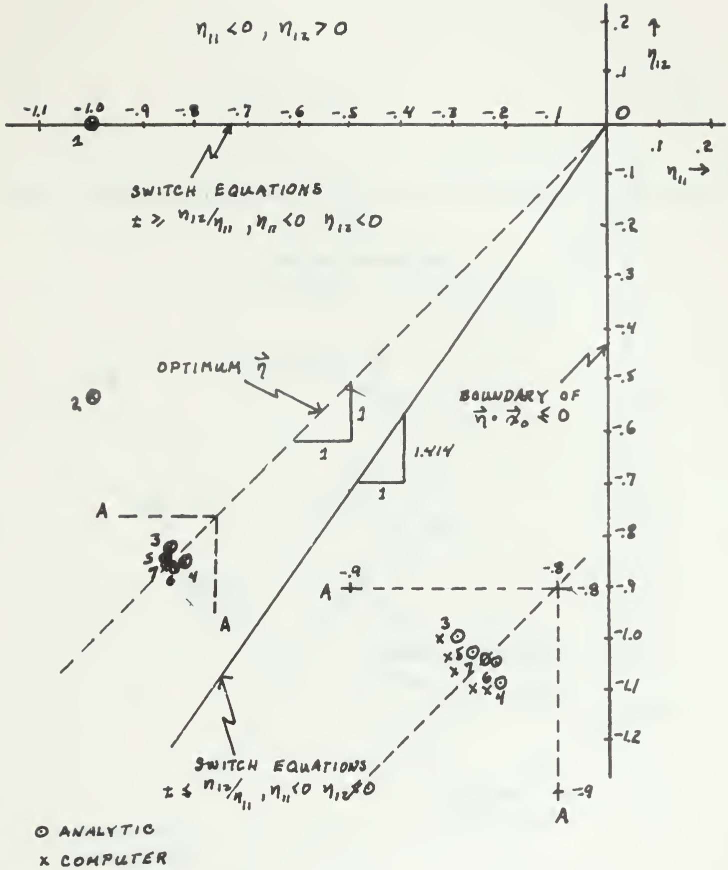


FIGURE V-5 PLOT OF $\eta_{\lambda 2}$ VERSUS $\eta_{\lambda 1}$
FOR $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \mu$, $\vec{x}_0 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$, $\Delta\tau = 0.375$

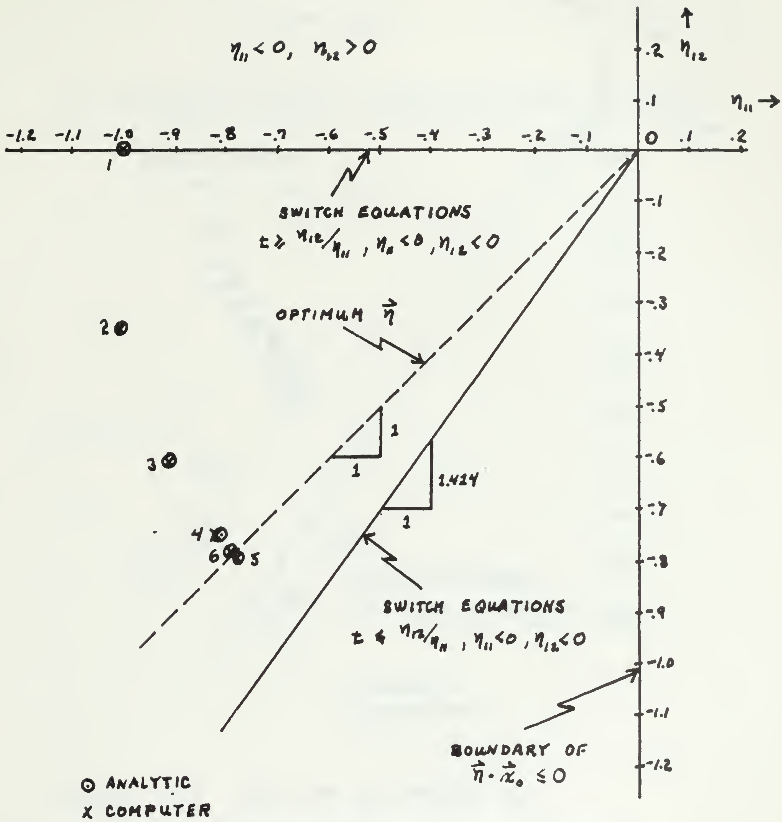


FIGURE V-6 PLOT OF η_{i2} VERSUS η_{i1}
FOR $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u \ ; \ \vec{x}_0 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \Delta\tau = 0.25$

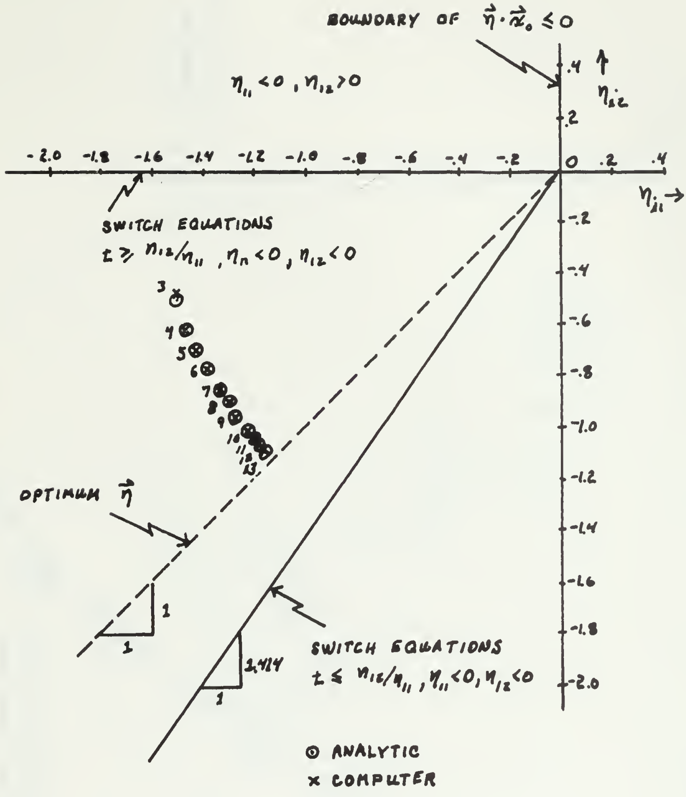


FIGURE V-7 PLOT OF η_{i2} VERSUS η_{i1}

FOR $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \mu$, $\vec{x}_0 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$, $\Delta\tau = 0.1$

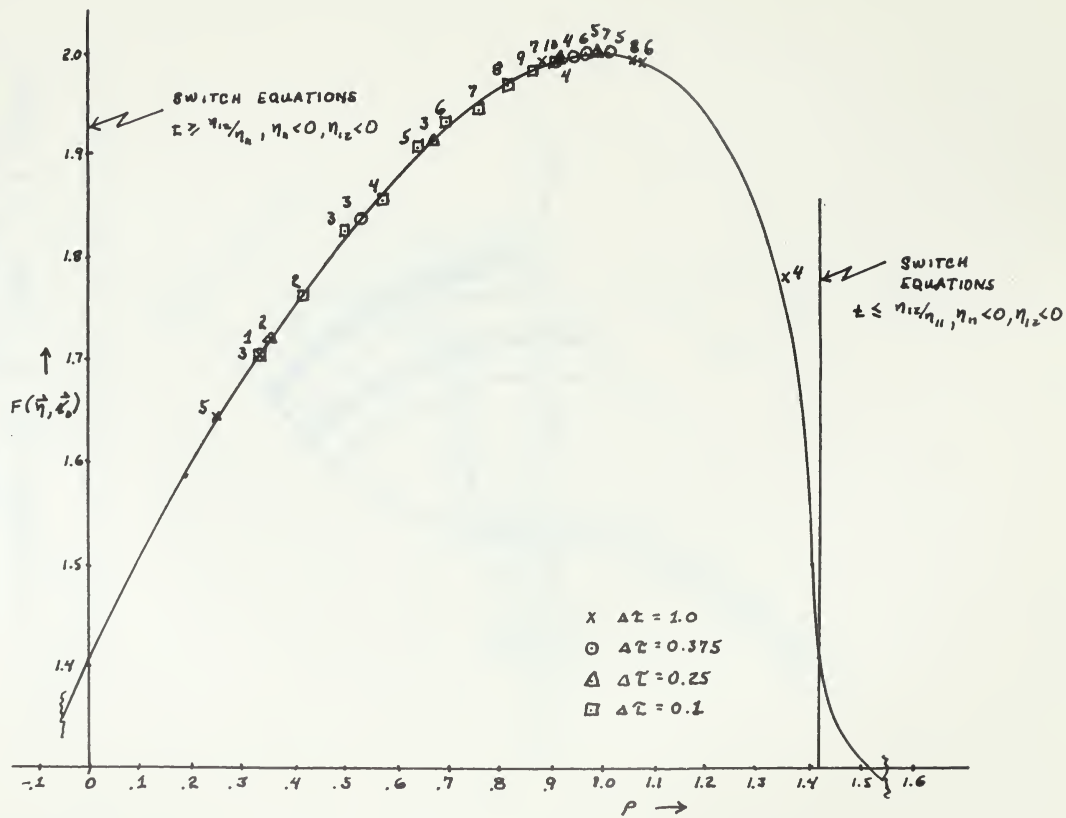


FIGURE V-8 STEEPEST ASCENT PATTERNS FOR DIFFERENT $\Delta\tau$

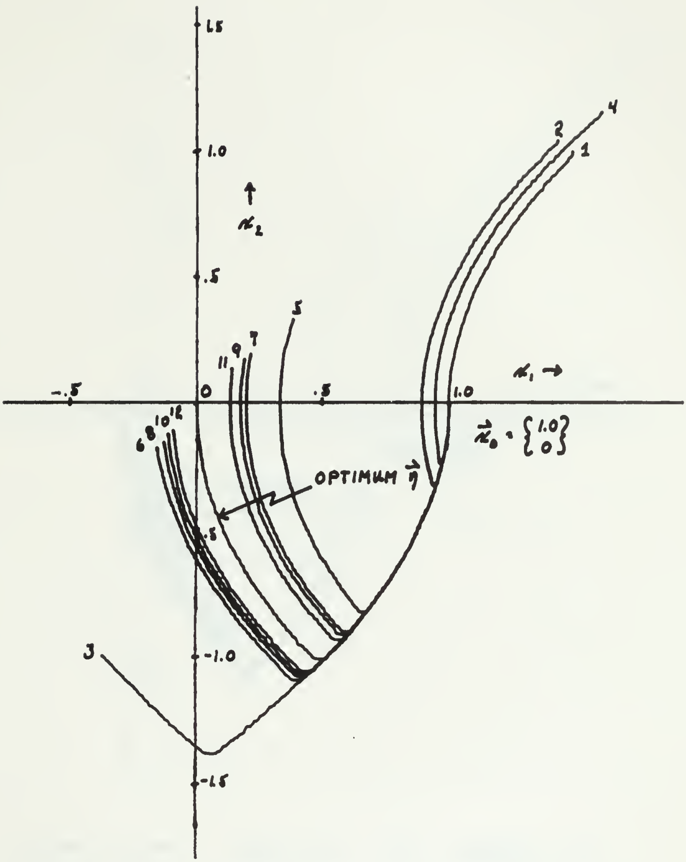


FIGURE V-9 PLOTS OF x_2 VERSUS x_1 FOR $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u$, $\vec{x}_0 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$, $\Delta\tau = 2.0$

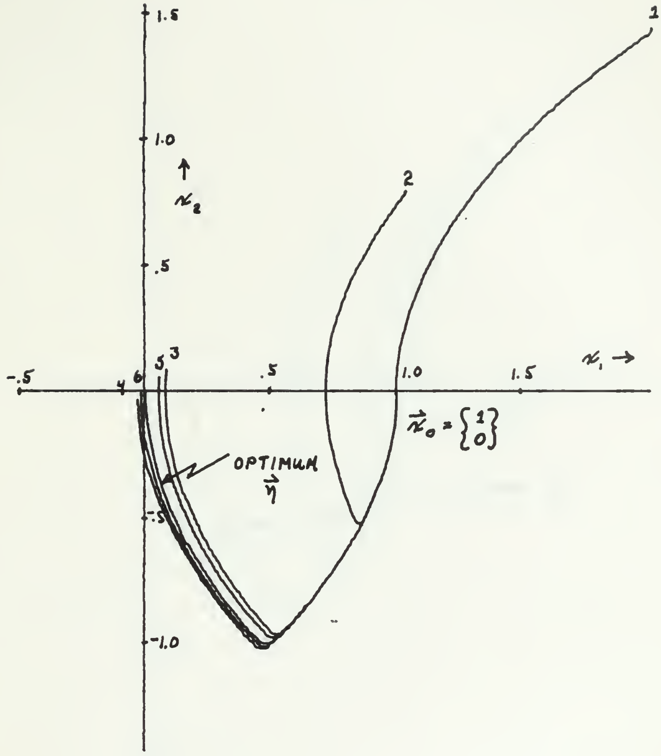


FIGURE V-10 PLOTS OF x_2 VERSUS x_1

FOR $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \mu$, $\vec{x}_0 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$, $\Delta\tau = 0.375$

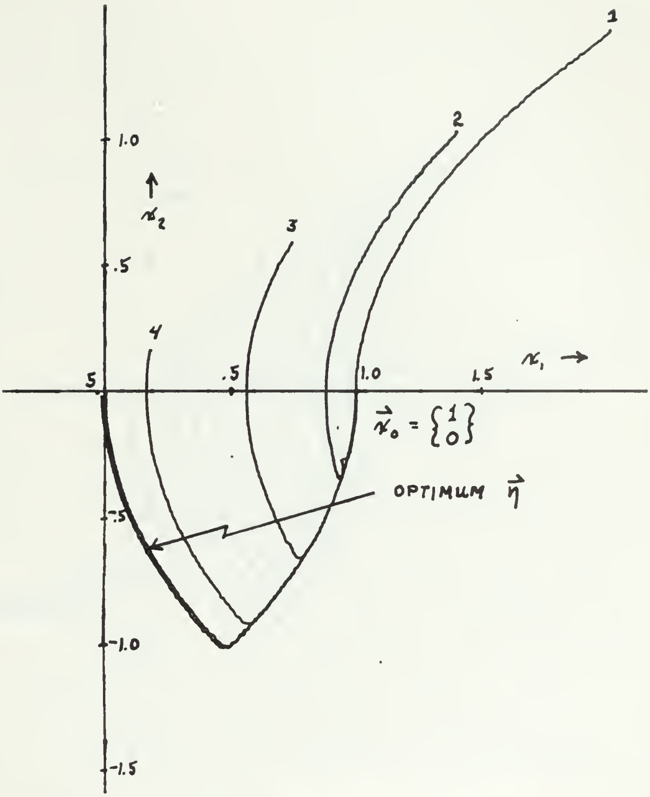


FIGURE V-11 PLOTS OF x_2 VERSUS x_1

FOR $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u$, $\vec{x}_0 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$, $\Delta\tau = 0.25$

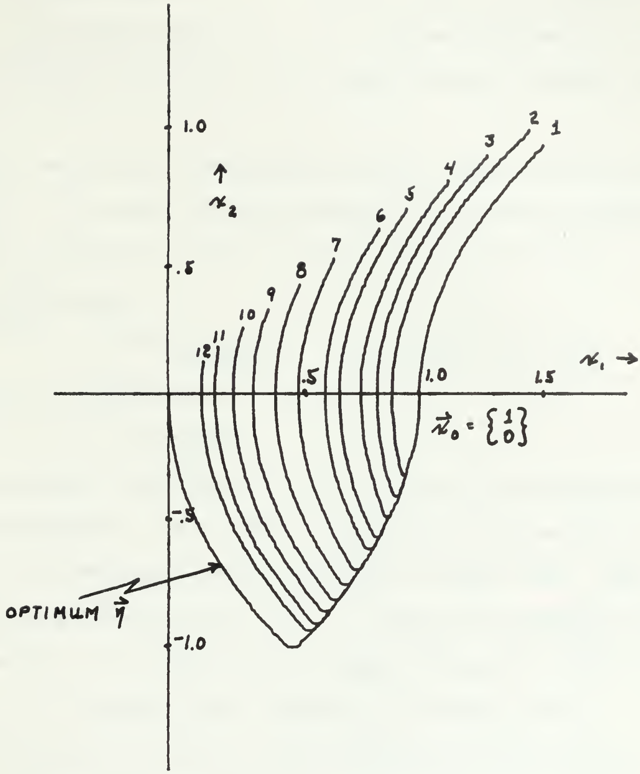


FIGURE V-12 PLOTS OF x_2 VERSUS x_1

FOR $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u$, $\vec{x}_0 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$, $\Delta\tau = 0.1$

V-12 respectively to indicate the relative error in $\vec{x}(t_1)$ for successive iterations of $\vec{\eta}$. The initial choice of $\vec{\eta}_1 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$ on the boundary of $\vec{\eta}_1 \cdot \vec{x}(0) \leq 0$ was chosen for $\Delta\tau = 1.0$, for $\Delta\tau = 0.25$ and 0.375, $\vec{\eta}_1 = \frac{-\vec{x}(0)}{|\vec{x}(0)|}$ was chosen and for $\Delta\tau = 0.1$, $\vec{\eta}_1$ was arbitrarily chosen. There was no apparent effect due to this limited choice of initial values of $\vec{\eta}_1$.

The plant was further investigated for another given initial state, $\vec{x}(0) = \begin{Bmatrix} 1 \\ -0.5 \end{Bmatrix}$. The plot of $F(\vec{\eta}, \vec{x}_0)$ versus ρ is presented in Figure V-13. Again the difference equation (V-13) was linearized about $\vec{\eta}^{\,o} = \begin{Bmatrix} -1.000 \\ -0.561 \end{Bmatrix}$ giving

(V-19) $$\vec{\delta} = \begin{bmatrix} (-0.668\,\Delta\tau + 1) & 1.198\,\Delta\tau \\ 1.142\,\Delta\tau & (-2.047\,\Delta\tau + 1) \end{bmatrix} d\vec{\eta}$$

so that A has the eigenvalues

(V-20) $$\lambda = 1,\ 1 - 2.715\,\Delta\tau$$

Analytic and computer computations for $\Delta\tau = 0.1$ and 1.0 were made and the data are presented in Table V-III and displayed in Figures V-14 and V-15. The hill climbing patterns are presented in Figure V-16. Again $\Delta\tau = 1.0$ gave oscillations while $\Delta\tau = 0.1$ gave slow convergence, and as before as $\vec{x}(t_1) \longrightarrow 0$, the correction vector, $-(\vec{z}(t_1, \vec{\eta}) + \vec{x}_0)\Delta\tau \longrightarrow 0$ independent of the choice of $\Delta\tau$. The plot $x_2(t)$ versus $x_1(t)$ is presented in Figure V-17 for $\Delta\tau = 1.0$ to display the relative error in $\vec{x}(t_1)$ for successive values of $\vec{\eta}$.

<u>Plant 2)</u>

The computer was set up as shown in Figure V-18 using the basic circuits of Section IV for

(V-21) $$\dot{\vec{x}} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u, \quad |u| \leq 1$$

Initial states of $\vec{x}(0) = \begin{Bmatrix} 0.2 \\ 0 \end{Bmatrix}$, $\begin{Bmatrix} 3.5 \\ 0 \end{Bmatrix}$, and $\begin{Bmatrix} 30 \\ 0 \end{Bmatrix}$ were investigated

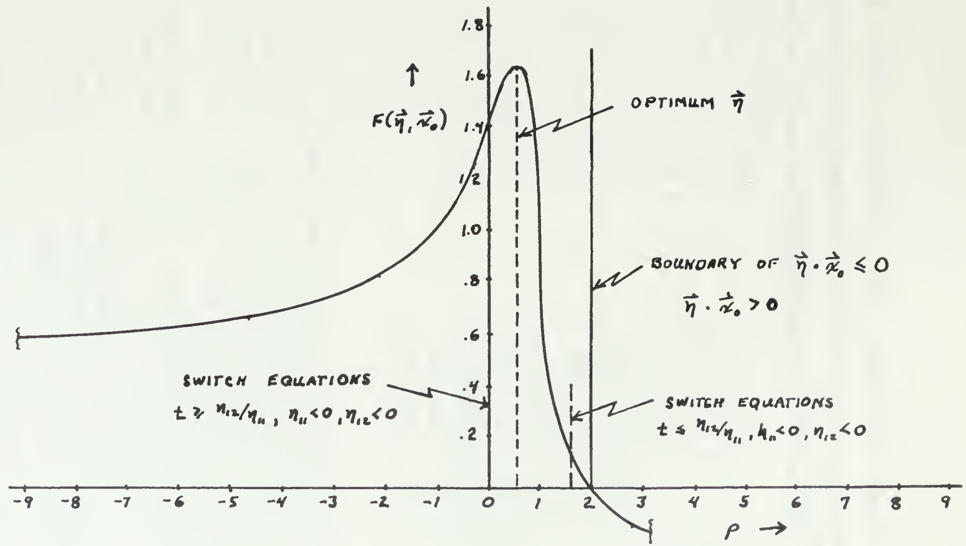


FIGURE V-13 $F(\vec{\eta}, \vec{x}_0)$ VERSUS ρ FOR $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \mu$, $\vec{x}_0 = \begin{Bmatrix} 1.0 \\ -.5 \end{Bmatrix}$

TABLE V-III ANALYTIC AND COMPUTER DATA

RUN	η_1	η_2	t_1	$x_1(t_1)$	$x_2(t_1)$	$y_1(t_1)$	$y_2(t_1)$	$d\eta_1/dt$	$d\eta_2/dt$	$\Delta\eta_1$	$\Delta\eta_2$	ρ
$\Delta\tau = 1.0$ ANALYTIC												
1	-1.326	-.0624	1.449			-1.040	+1.350	+.040	-.850	+.040	-.850	
2	-1.286	-.914	1.593			-.767	+.177	-.233	+.323	-.233	+.323	
3	-1.519	-.589	1.597			-1.121	+.821	+.121	-.321	+.121	-.321	
4	-1.398	-.910	1.611			-.876	±.309	-.124	+.191	-.124	-.191	
5	-1.522	-.719	1.612			-1.078	+.668	+.078	-.168	+.078	-.168	
6	-1.444	-.887	1.618			-.933	+.390	-.067	+.110	-.067	+.110	
7	-1.511	-.777	1.619			-1.055	+.609	+.055	-.109	+.055	-.109	
8	-1.456	-.886	1.619			-.940	+.401	-.060	+.099	-.060	+.099	
9	-1.516	-.787	1.620			-1.042	+.580	+.042	-.080	+.042	-.080	
10	-1.474	-.867	1.620			-.966	+.444	-.034	+.056	-.034	+.056	
11	-1.508	-.811	1.620			-1.022	+.542	+.022	-.042	+.022	-.042	
12	-1.486	-.853										
$\Delta\tau = 1.0$ COMPUTER												
1	-1.326	-.0624	1.460	+1.224	+.818	-1.052	+1.356	+.052	-.856	+.052	-.856	+.047
2	-1.273	-.918	1.619	-.296	-.334	-.752	+.145	-.248	+.355	-.248	+.385	+.721
3	-1.522	-.563	1.603	+.412	+.346	-1.144	+.849	+.144	-.359	+.144	-.359	+.370
4	-1.378	-.922	1.609	-.196	-.212	-.844	+.266	-.156	+.234	-.156	+.234	+.671
5	-1.533	-.688	1.619	+.251	+.210	-1.086	+.710	+.086	-.210	+.086	-.210	+.449
6	-1.448	-.898	1.633	-.098	-.115	-.922	+.373	-.078	+.128	-.078	+.128	+.621
7	-1.525	-.771	1.633	+.1179	+.1048	-1.052	+.606	+.052	-.106	+.052	-.106	+.506
8	-1.474	-.876	1.632	-.0968	-.0784	-.964	+.424	-.036	+.076	-.036	+.076	+.594
9	-1.510	-.801	1.630	-.0428	-.0526	-1.038	+.556	+.038	-.056	+.038	-.056	+.531
10	-1.472	-.856	1.633	+.0686	+.0528	-.979	+.451	-.021	+.049	-.021	+.049	+.581
11	-1.493	-.807	1.631	-.0159	-.0300	-1.035	+.534	+.035	-.034	+.035	-.034	+.539
12	-1.458	-.841										+.576

TABLE V-III (CONTINUED)

RUN	η_1	η_2	t_1	$x_1(t_1)$	$x_2(t_1)$	$y_1(t_1)$	$y_2(t_1)$	$d\eta_1/dt$	$d\eta_2/dt$	$\Delta\eta_1$	$\Delta\eta_2$	P
$\Delta\tau$ = 0.1 ANALYTIC												
1	USED	DATA	FOR	$\Delta\tau$ =	1.0							
2	-1.286	-.914	1.596			-.766	+.174	-.234	+.326	-.023	+.033	.711
3	-1.309	-.881	1.608			-.838	+.262	-.162	+.238	-.016	+.024	.673
4	-1.325	-.857	1.613			-.882	+.319	-.118	+.181	-.012	+.018	.647
5	-1.337	-.839	1.617			-.913	+.361	-.097	+.139	-.009	+.014	.628
6	-1.346	-.825	1.614			-.918	+.386	-.082	+.114	-.008	+.011	.614
7	-1.354	-.814	1.621			-.953	+.419	-.047	+.081	-.005	+.008	.601
8	-1.359	-.806	1.619			-.958	+.431	-.042	+.069	-.004	+.007	.594
9	-1.363	-.799	1.620			-.967	+.448	-.033	+.052	-.003	+.005	.586
10	-1.366	-.794	1.621			-.975	+.459	-.025	+.041	-.002	+.004	.581
11	-1.368	-.790	1.621			-.980	+.467	-.020	+.033	-.002	+.003	.577
12	-1.370	-.787										
$\Delta\tau$ = 0.1 COMPUTER												
1	USED	DATA	FOR	$\Delta\tau$=1.0								
2	-1.274	-.981	1.619	-.296	-.334	-.752	+.145	-.248	+.355	-.025	+.035	+.721
3	-1.298	-.883	1.619	-.266	-.265	-.836	+.239	-.164	+.261	-.016	+.026	+.680
4	-1.315	-.857	1.629	-.213	-.200	-.886	+.302	-.114	+.198	-.011	+.020	+.652
5	-1.326	-.837	1.631	-.176	-.157	-.921	+.364	-.079	+.154	-.008	+.015	+.632
6	-1.334	-.821	1.629	-.140	-.123	-.932	+.380	-.068	+.120	-.007	+.012	+.616
7	-1.341	-.809	1.629	-.115	-.097	-.950	+.406	-.050	+.094	-.005	+.009	+.603
8	-1.346	-.800	1.629	-.100	-.078	-.961	+.425	-.039	+.075	-.004	+.008	+.593
9	-1.350	-.792	1.632	-.080	-.062	-.972	+.441	-.028	+.059	-.003	+.006	+.588
10	-1.352	-.787	1.632	-.070	-.052	-.976	+.451	-.024	+.049	-.002	+.005	+.582
11	-1.355	-.782	1.632	-.059	-.042	-.983	+.460	-.017	+.040	-.002	+.004	+.577
12	-1.357	-.778										+.574

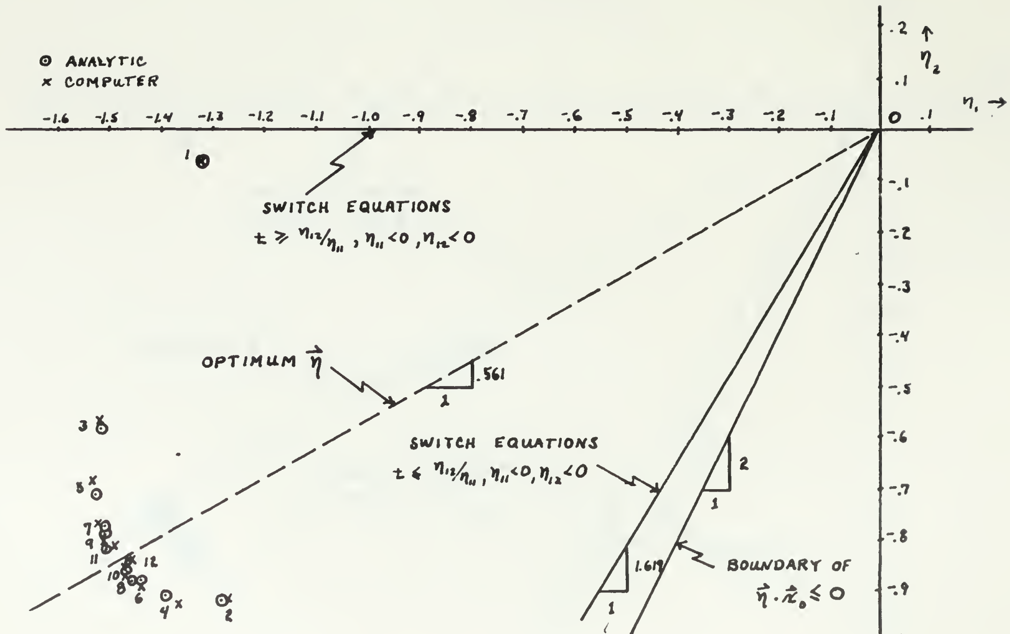


FIGURE V-14 PLOT OF η_{i2} VERSUS η_{i1} FOR $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u$, $\vec{x}_0 = \begin{Bmatrix} 1.0 \\ -.5 \end{Bmatrix}$, $\Delta\tau = 1.0$

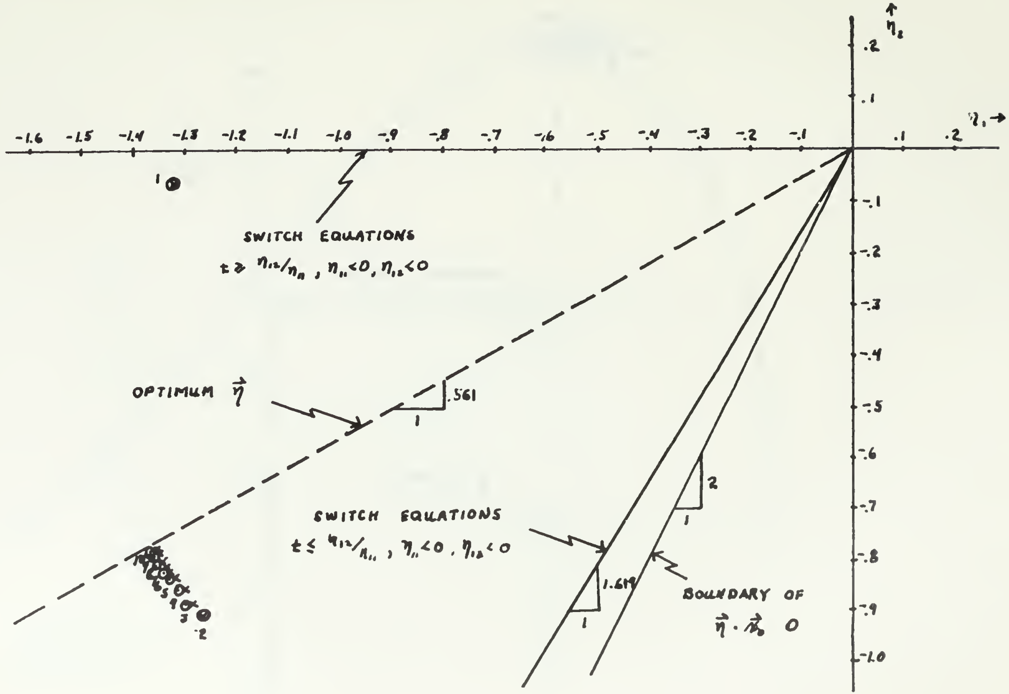


FIGURE V-15 PLOT OF η_{12} VERSUS η_{11} FOR $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \mu$, $\vec{x}_0 = \begin{Bmatrix} 1.0 \\ -.5 \end{Bmatrix}$, $\Delta\tau = 0.1$

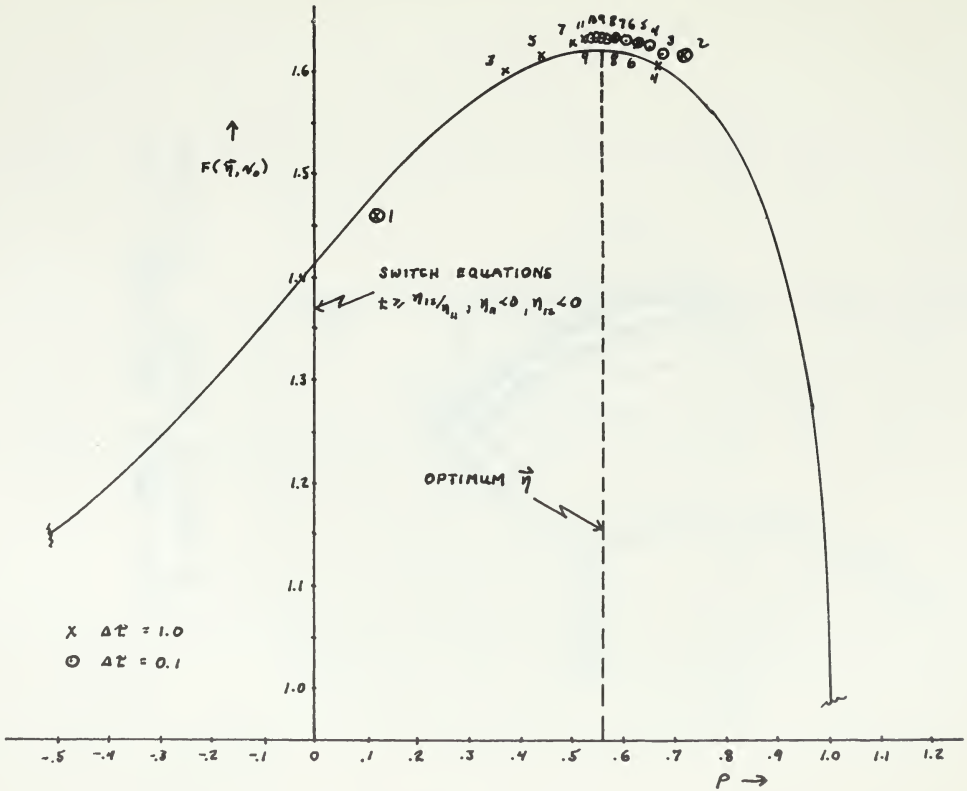


FIGURE V-16 STEEPEST ASCENT PATTERNS FOR DIFFERENT VALUES OF $\Delta\tau$

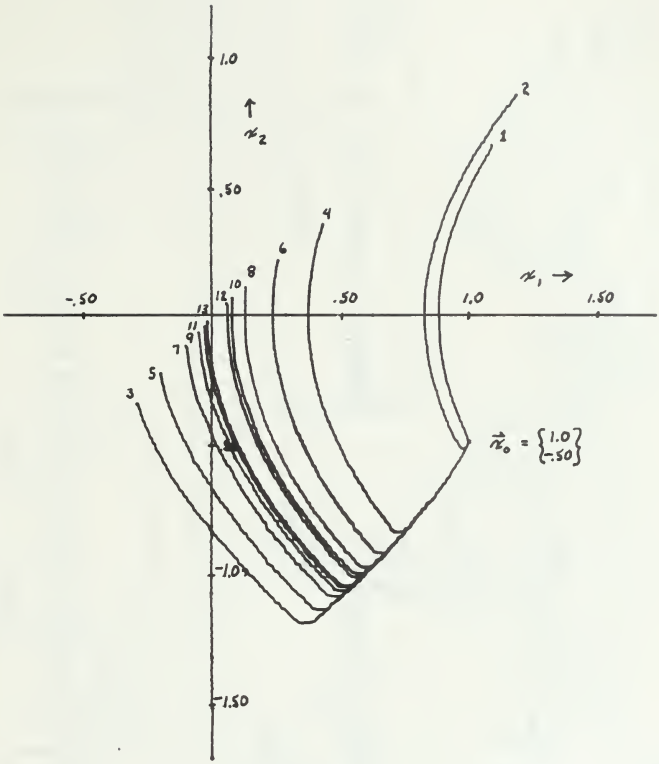


FIGURE V-17 PLOTS OF x_2 VERSUS x_1

FOR $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u$, $\vec{x}_0 = \begin{Bmatrix} 1.0 \\ -.50 \end{Bmatrix}$, $\Delta T = 1.0$

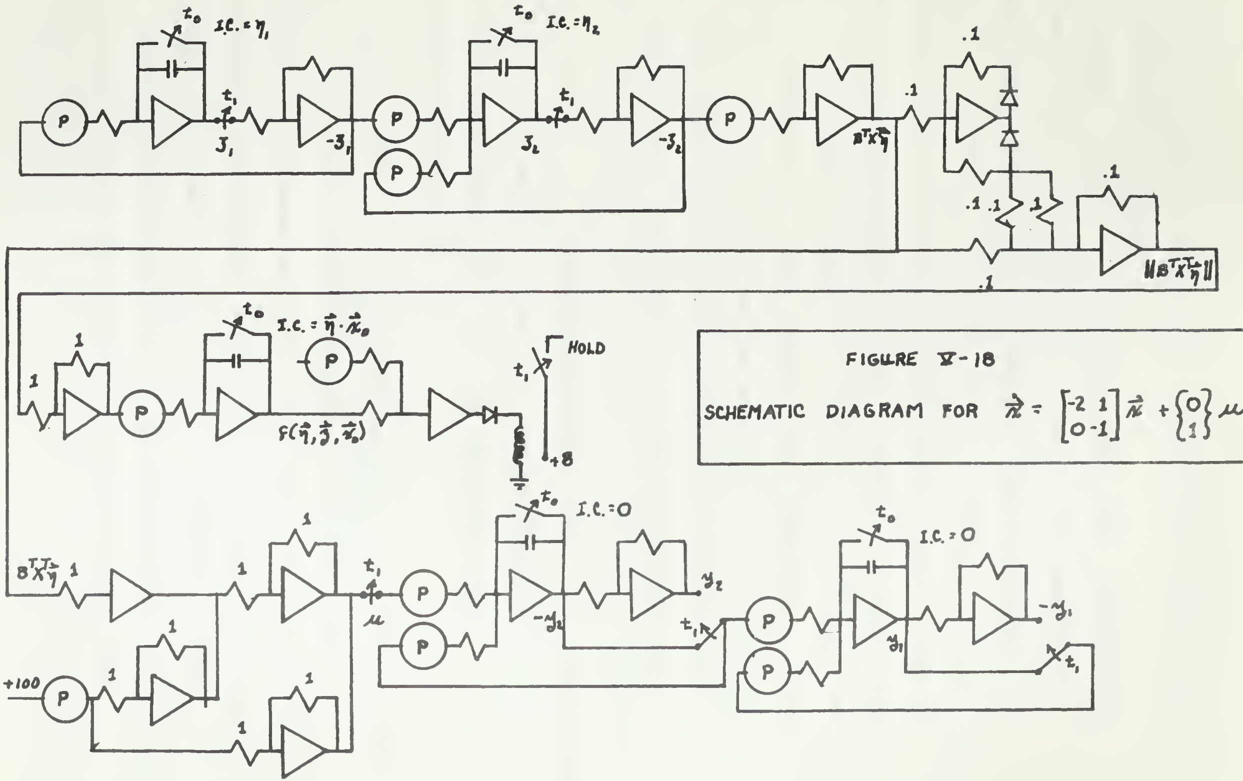


FIGURE V-18

SCHEMATIC DIAGRAM FOR $\dot{\vec{x}} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \mu$

to determine the effect of computing solutions through different multiples of the system time constant.

For $\vec{x}(0) = \begin{Bmatrix} 0.2 \\ 0 \end{Bmatrix}$, $\Delta\tau = 0.5$ and 1.0 were used. The data are tabulated in Table V-IV. Plots of η_{i2} versus η_{i1} are presented in Figures V-19 and V-20, the $x_2(t)$ versus $x_1(t)$ plots are presented in Figures V-21 and V-22, and the plots of $y_2(t)$ versus $y_1(t)$ and $w_2(t)$ versus $w_1(t)$ are presented in Figures V-23 and V-24 to indicate the range of $\vec{y}(t)$ and $\vec{w}(t)$ in the computation of $\vec{z}(t_1)$. As was the case with Plant 1), $\Delta\tau = 1.0$ gave some oscillations and relatively slow convergence, whereas $\Delta\tau = 0.5$ gave much more rapid convergence and eliminated much of the oscillations. Again it was noted that as $\vec{x}(t_1) \longrightarrow 0$, the correction vector $-(\vec{z}(t_1, \vec{\eta}) + \vec{x}_0)\Delta\tau \longrightarrow 0$ independent of the choice of $\Delta\tau$.

For $\vec{x}(0) = \begin{Bmatrix} 3.5 \\ 0 \end{Bmatrix}$, the choice of $\Delta\tau$ was dependent upon the criterion $|\Delta\vec{\eta}_{i+1}| < |\Delta\vec{\eta}_i|$. The data are tabulated in Table V-IV, the η_{i2} versus η_{i1} plot is presented in Figure V-26. Using this criterion for the selection $\Delta\tau$ rapid convergence with some oscillation was obtained. Again as the state vector $\vec{x}(t_1) \longrightarrow 0$, the correction vector $-(\vec{z}(t_1, \vec{\eta}) + \vec{x}_0)\Delta\tau \longrightarrow 0$.

For $\vec{x}(0) = \begin{Bmatrix} 30 \\ 0 \end{Bmatrix}$ the choice of $\Delta\tau$ was dependent upon the criterion $\Delta\vec{\eta}_{i+1} \cdot \Delta\vec{\eta}_i > 0$. The data are presented in Table V-IV, the η_{i2} versus η_{i1} plot is presented in Figure V-27, the $x_2(t)$ versus $x_1(t)$ plot is presented in Figure V-28 and the plots of $y_2(t)$ versus $y_1(t)$ and $w_2(t)$ versus $w_1(t)$ are presented in Figure V-29. It is noted from Table V-IV that the additional requirement that $\vec{\eta} \cdot \vec{x}_0 \leqslant 0$ was used in the selection of $\Delta\tau$, hence using the finite difference equation it does become possible for $\vec{\eta}$ to leave the domain. In Figure V-29 it is noted that the

RUN	η_1	η_2	t	$x_1(t_1)$	$x_2(t_1)$	$p_1(t_1)$	$p_2(t_1)$	$d\eta_1/d\tau$	$d\eta_2/d\tau$	$\Delta\tau$	$\Delta\eta_1$	$\Delta\eta_2$
$\vec{x}_0 = \begin{Bmatrix} .2 \\ 0 \end{Bmatrix}$ $\Delta\tau = 1.0$												
1	-1.0	0	.4960	+.1499	+.3839	-.1910	+.6219	-.0090	-.6219	1.0	-.0090	-.6219
2	-1.009	-.622	.5950	-.0245	-.2209	-.0980	-.3960	-.1020	+.3960	1.0	-.1020	+.3960
3	-1.111	-.226	.5964	+.0726	+.2173	-.1604	+.3882	-.0396	-.3882	1.0	-.0396	-.3882
4	-1.151	-.614	.6379	-.0178	-.0941	+.0078	-.1760	-.2078	+.1760	1.0	-.2078	+.1760
5	-1.353	-.438	.6311	+.0342	+.1173	-.2391	+.2174	+.0391	-.2174	1.0	+.0391	-.2174
6	-1.314	-.655	.6430	-.0110	-.0528	-.1479	-.0982	-.0521	+.0982	1.0	-.0521	+.0982
7	-1.366	-.557	.6429	+.0112	+.0428	-.2096	+.0805	+.0096	-.0805	1.0	+.0096	-.0805
8	-1.357	-.638	.6423	-.0046	-.0216	-.1744	-.0418	-.0256	+.0418	1.0	-.0256	+.0418
9	-1.382	-.596	.6405	+.0050	+.0203	-.2109	+.0390	+.0109	-.0390	1.0	+.0109	-.0390
$\vec{x}_0 = \begin{Bmatrix} .2 \\ 0 \end{Bmatrix}$ $\Delta\tau = 0.5$												
1	-1.0	0	.4945	+.1496	+.3863	-.1906	+.6262	-.0094	-.6262	0.5	-.0047	-.3131
2	-1.005	-.313	.6269	+.0378	+.1285	-.2646	+.2370	+.0646	-.2370	0.5	+.0323	-.1185
3	-.9624	-.432	.6444	-.0011	-.0060	-.1892	-.0096	-.0108	+.0096	0.5	-.0054	+.0048
4	-.9678	-.427	.6449	+.0021	+.0081	-.2009	+.0153	+.0009	-.0155	0.5	+.0004	-.0057
5	-.9674	-.432	.6449	+.0007	+.0021	-.1940	+.0016	-.0060	-.0016	0.5	-.0030	-.0008
$\vec{x}_0 = \begin{Bmatrix} 3.5 \\ 0 \end{Bmatrix}$												
1	-1.0	0	1.284	+.5266	+.7355	-3.612	+2.706	+.112	-2.706	0.5	+.056	-1.353
2	-.944	-1.353	1.533	+.0777	+.1644	-4.579	+.792	+1.079	-.792	0.25	+.269	-.198
3	-.675	-1.551	1.522	-.0643	-.2190	-1.179	-1.015	-2.321	+1.015	0.25	-.580	+.256
4	-1.255	-1.295	1.489	+.1620	+.3009	-4.835	+1.330	+1.335	-1.330	0.25	+.334	-.332
5	-.921	-1.627	1.549	+.0065	+.0242	-3.670	+.0929	+.170	-.0929	0.125	+.021	-.011
6	-.900	-1.638	1.549	-.0013	+.0055	-3.504	+.0049	+.004	-.0049			
$\vec{x}_0 = \begin{Bmatrix} 30 \\ 0 \end{Bmatrix}$												
1	-1.0	0	2.169	+.774	+.890	-32.69	+7.94	+2.69	-7.94	.0625	+.168	-.496
2	-.832	-.496	2.232	+.627	+.779	-37.03	+7.350	+7.03	-7.35	.0156	+.112	-.123
3	-.720	-.619	2.263	+.561	+.721	-39.90	+7.06	+9.90	-7.06	.0156	+.154	-.110
4	-.566	-.729	2.300	+.484	+.649	-43.19	+6.60	+13.19	-6.60	.0156	+.206	-.103
5	-.360	-.832	2.381	+.316	+.503	-51.19	+5.59	+21.19	-5.59	.0078	+.165	-.043
6	-.195	-.875	2.480	+.381	+.067	-48.49	+2.33	+18.49	-2.33	.0039	+.072	-.009
7	-.123	-.884	2.463	-.024	+.019	-32.49	+.379	+2.49	-.379	.0020	+.006	-.001
8	-.153	-.881	2.463	.000	+.005	-31.42	+.279	+1.42	-.279	.0020	+.003	-.001

TABLE V - IV COMPUTER DATA FOR $\dot{\vec{x}} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u$

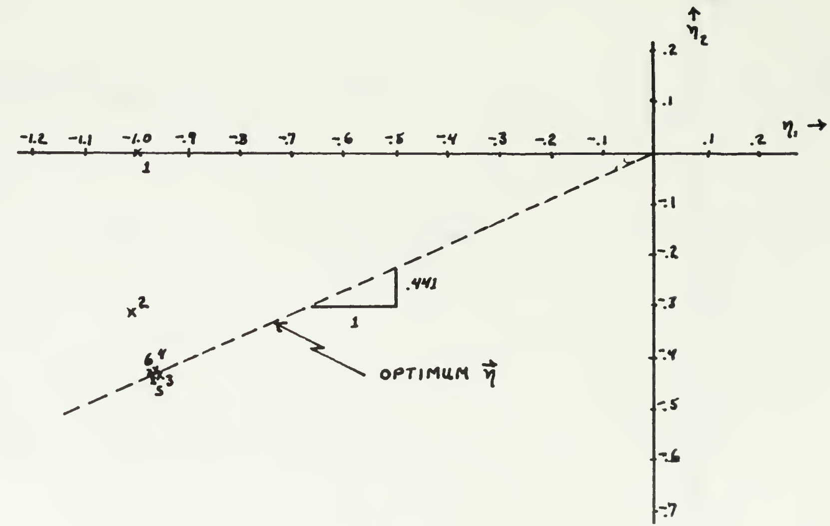


FIGURE V-19

PLOT OF η_{22} VERSUS η_{21} FOR $\dot{\vec{x}} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u$, $\vec{x}_0 = \begin{Bmatrix} .2 \\ 0 \end{Bmatrix}$, $\Delta\tau = 0.5$

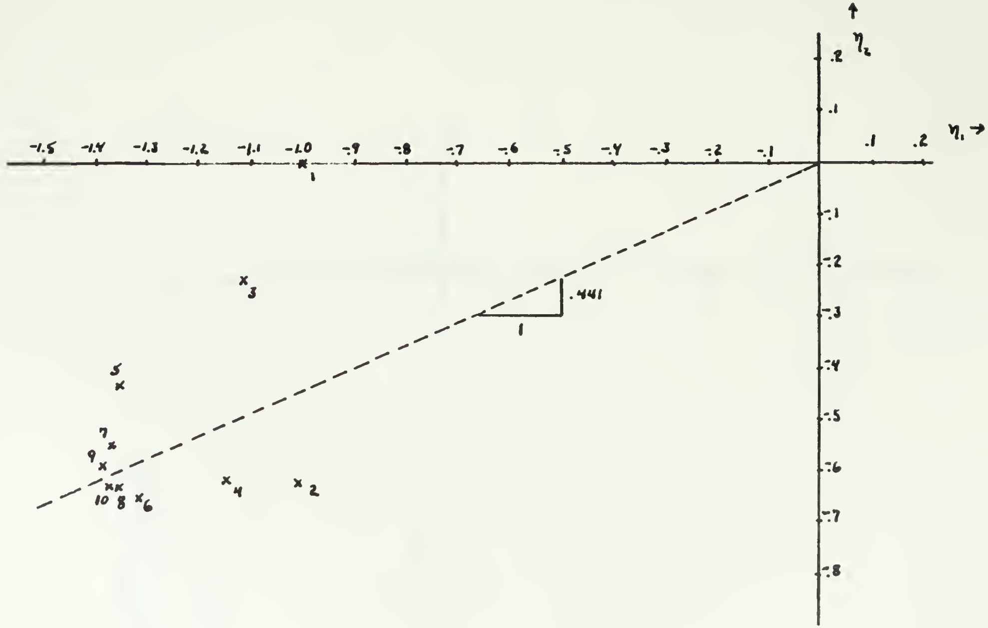


FIGURE V-20

PLOT OF η_{n2} VERSUS η_{n1} FOR $\dot{\vec{x}} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u$, $\vec{x}_0 = \begin{Bmatrix} .2 \\ 0 \end{Bmatrix}$, $\Delta T = 1.0$

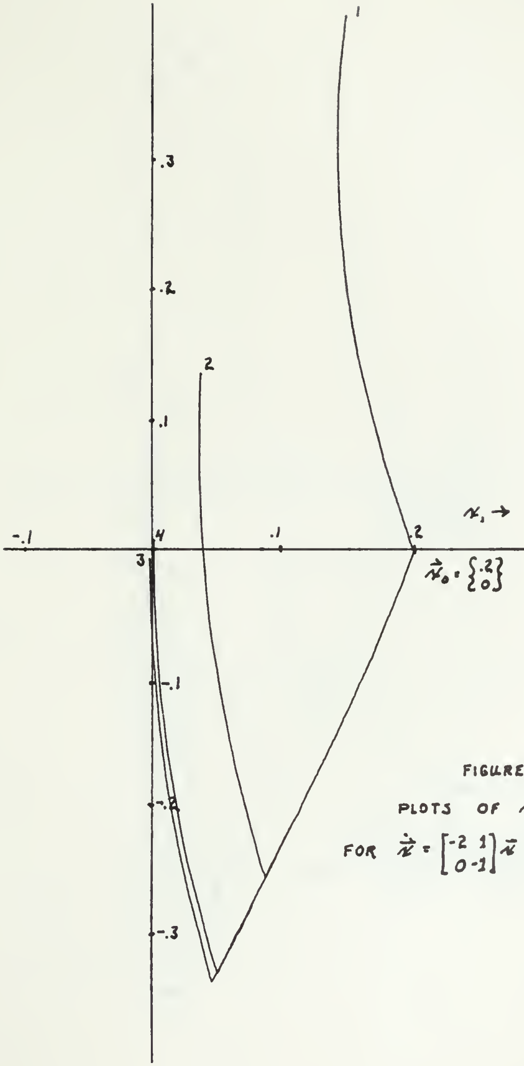


FIGURE V-21

PLOTS OF x_2 VERSUS x_1

FOR $\dot{\vec{x}} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u$, $\vec{x}_0 = \begin{Bmatrix} .2 \\ 0 \end{Bmatrix}$, $\Delta T = 0.5$

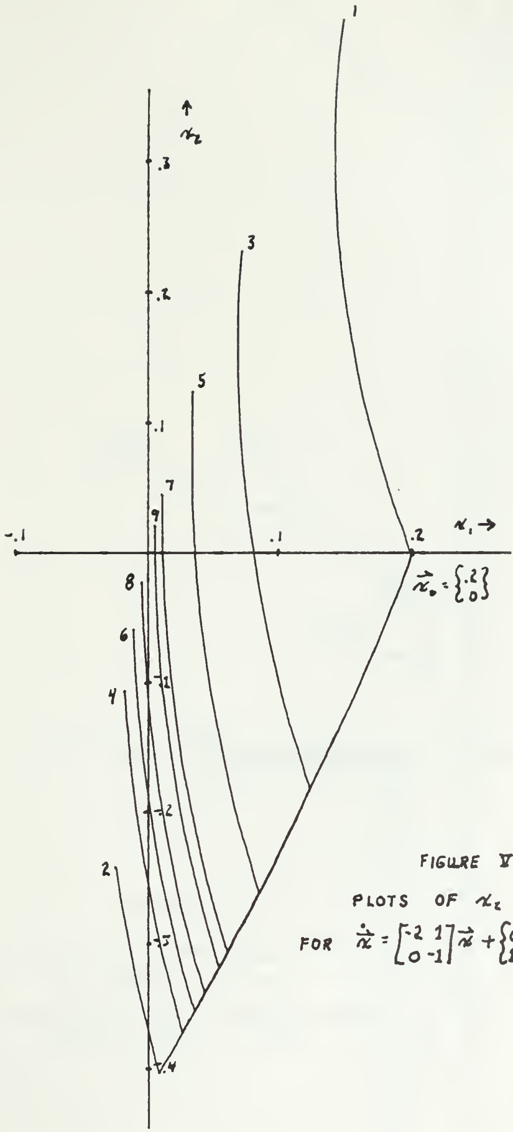


FIGURE V-22

PLOTS OF x_2 VERSUS x_1

FOR $\dot{\vec{x}} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u$, $\vec{x}_0 = \begin{Bmatrix} .2 \\ 0 \end{Bmatrix}$, $\Delta t = 1.0$

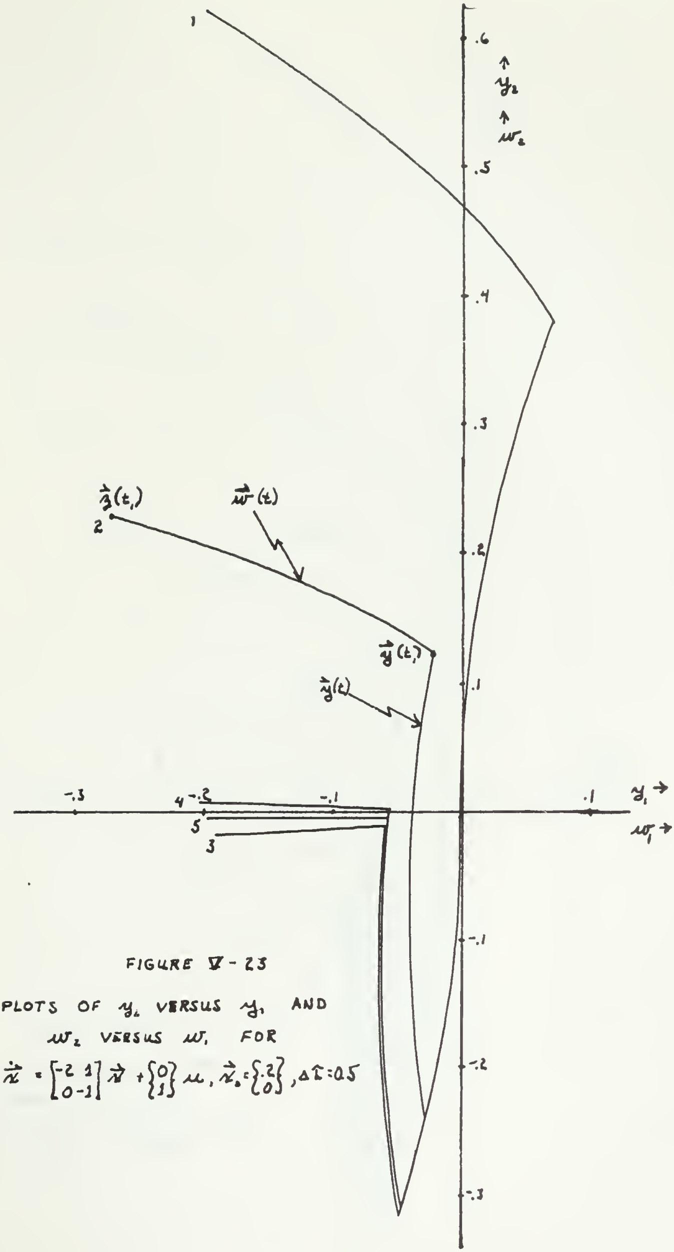


FIGURE V - 23

PLOTS OF y_2 VERSUS y_1 AND w_2 VERSUS w_1 FOR

$\dot{\vec{x}} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u, \ \vec{x}_0 = \begin{Bmatrix} .2 \\ 0 \end{Bmatrix}, \Delta\tau = 0.5$

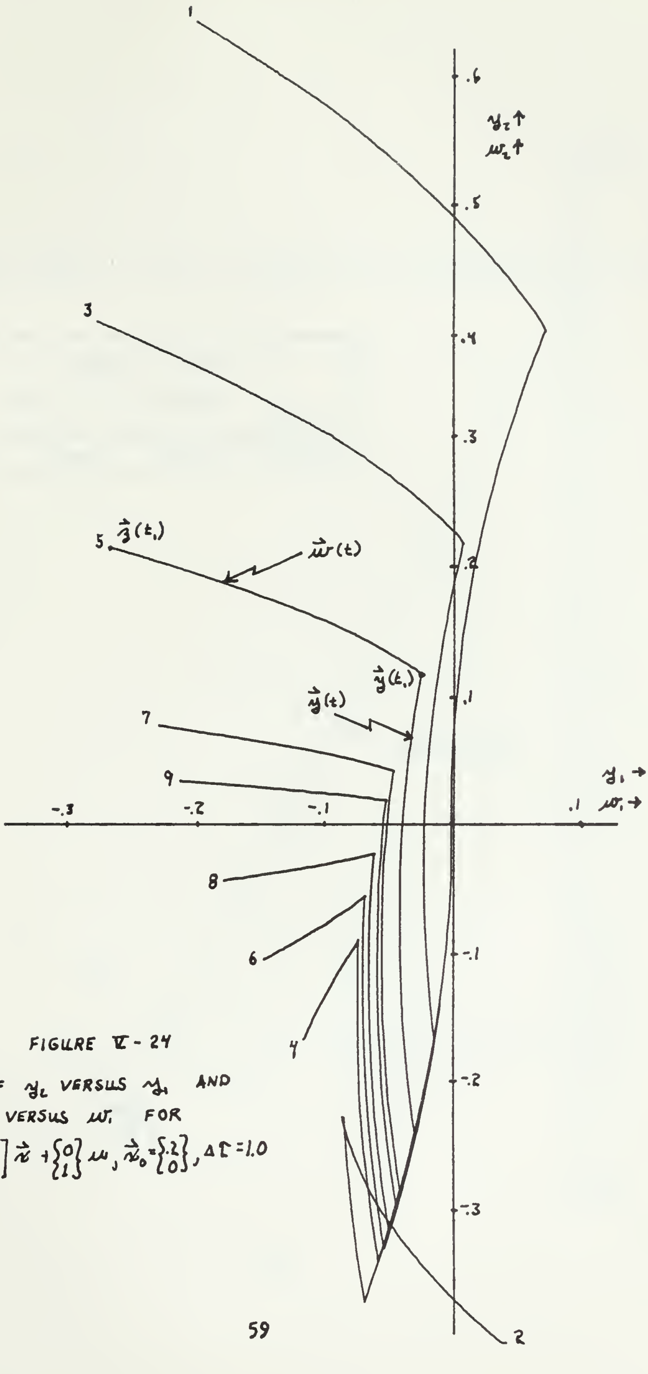


FIGURE V - 24

PLOTS OF y_2 VERSUS y_1 AND w_2 VERSUS w_1 FOR

$$\dot{\vec{x}} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u, \quad \vec{x}_0 = \begin{Bmatrix} .2 \\ 0 \end{Bmatrix}, \quad \Delta t = 1.0$$

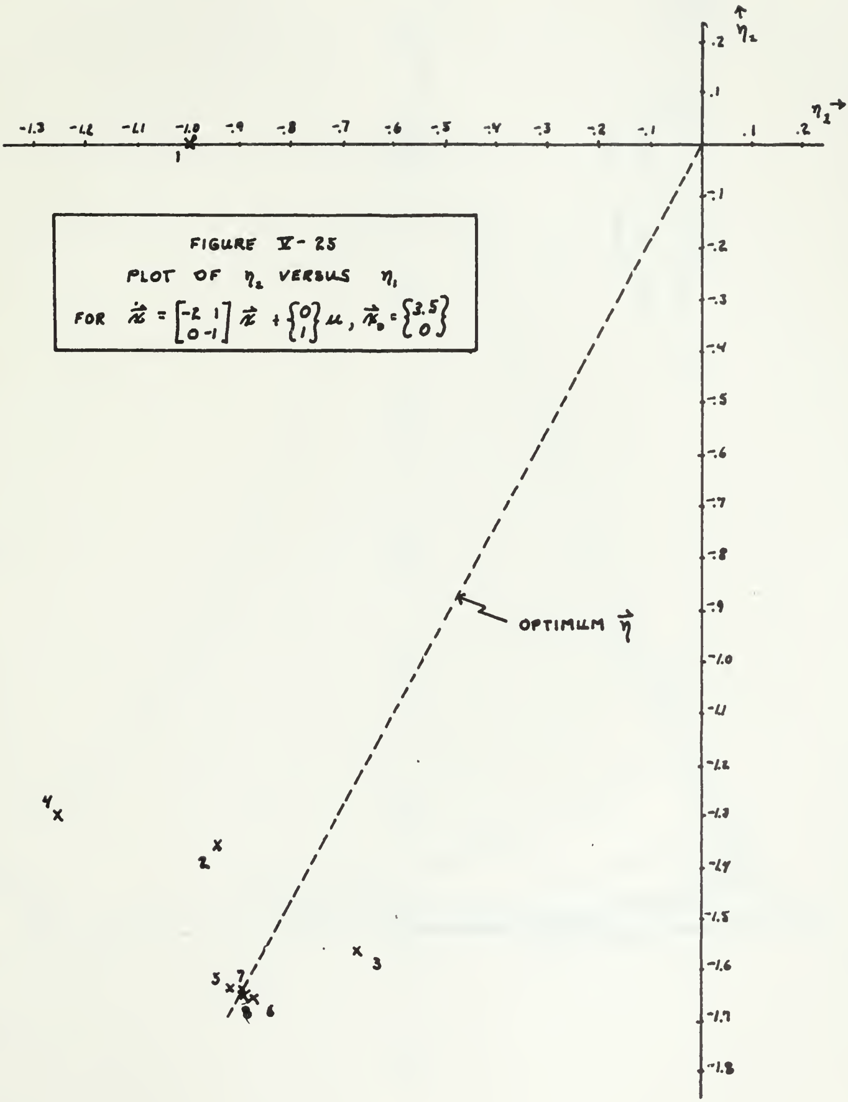


FIGURE V-25

PLOT OF η_2 VERSUS η_1

FOR $\dot{\vec{x}} = \begin{bmatrix}-2 & 1\\ 0 & -1\end{bmatrix}\vec{x} + \begin{Bmatrix}0\\ 1\end{Bmatrix}u$, $\vec{x}_0 = \begin{Bmatrix}3.5\\ 0\end{Bmatrix}$

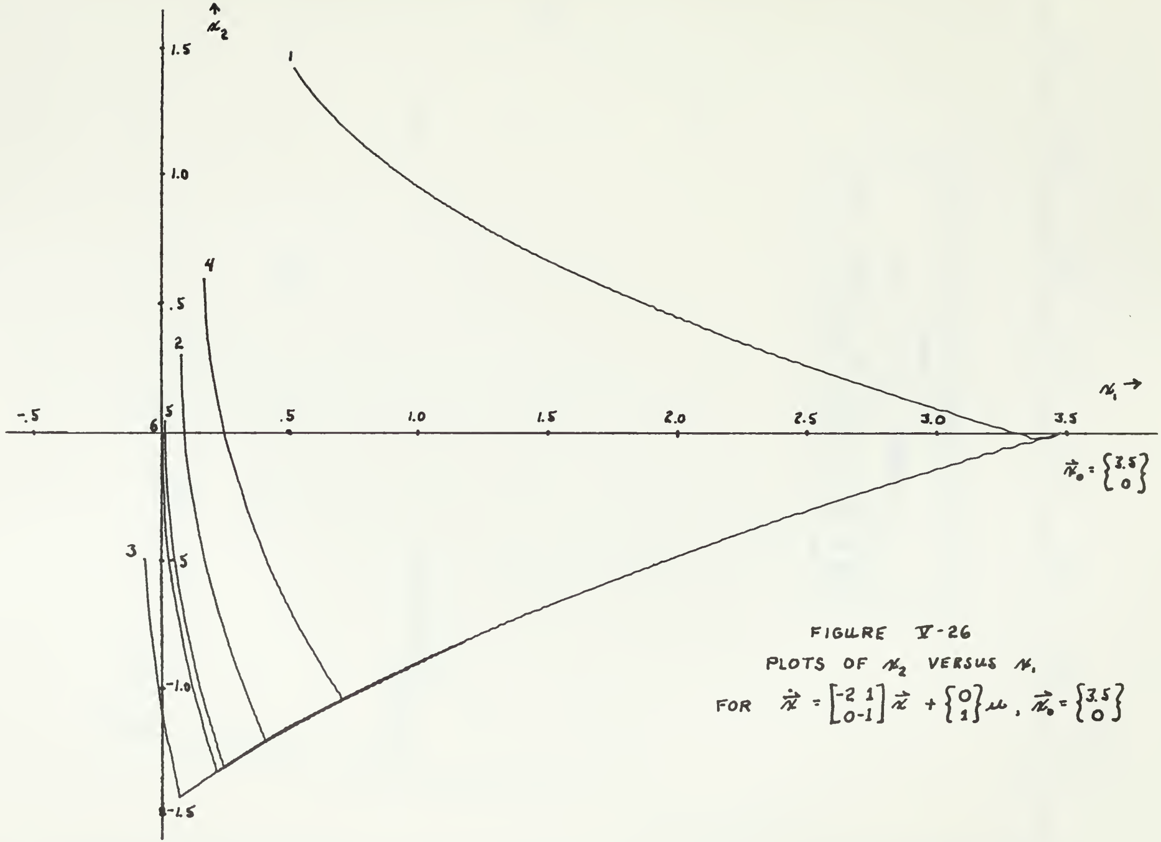


FIGURE V-26

PLOTS OF x_2 VERSUS x_1

FOR $\dot{\vec{x}} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u$, $\vec{x}_0 = \begin{Bmatrix} 3.5 \\ 0 \end{Bmatrix}$

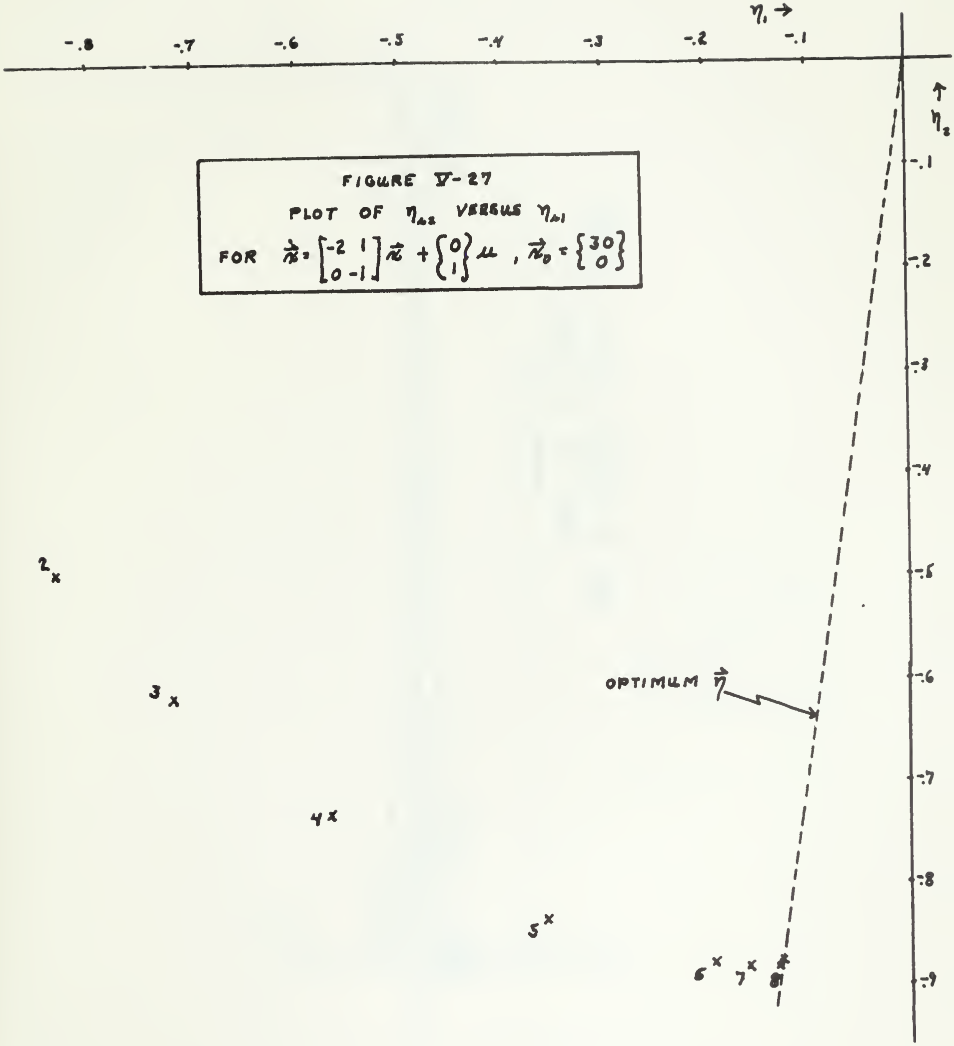


FIGURE V-27

PLOT OF η_{a2} VERSUS η_{a1}

FOR $\dot{\vec{x}} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u$, $\vec{x}_0 = \begin{Bmatrix} 30 \\ 0 \end{Bmatrix}$

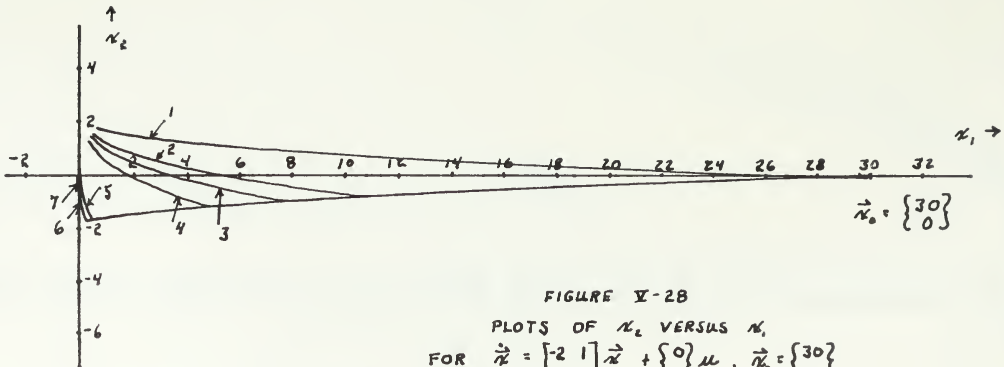


FIGURE V-28

PLOTS OF x_2 VERSUS x_1

FOR $\dot{\vec{x}} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u$, $\vec{x}_0 = \begin{Bmatrix} 30 \\ 0 \end{Bmatrix}$

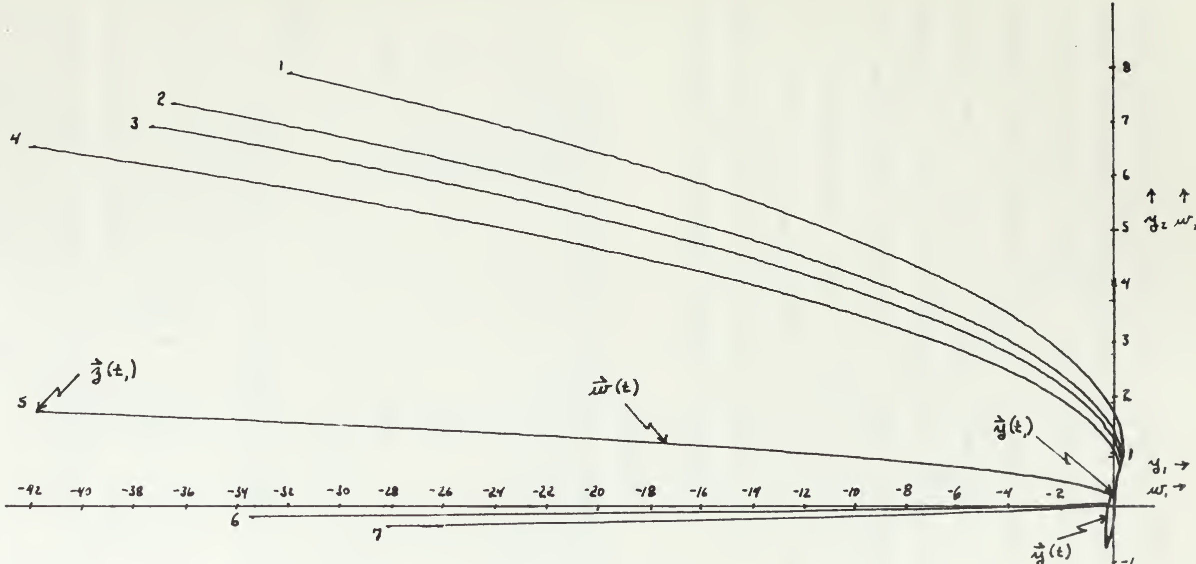


FIGURE V-29

PLOTS OF

y_2 VERSUS y_1

AND

w_2 VERSUS w_1

FOR

$$\dot{\vec{x}} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} w \, , \quad \vec{x}_0 = \begin{Bmatrix} 30 \\ 0 \end{Bmatrix}$$

range of $\vec{y}(t)$ is much smaller than the range of $\vec{w}(t)$. Any noise in the system would have a more pronounced effect on $\vec{y}(t)$ resulting in errors in $\vec{y}(t_1)$ and hence in $\vec{z}(t_1)$. This could possibly explain why $-(\vec{z}(t_1,\vec{\eta}) + \vec{x}_0) \not\rightarrow 0$ as $\vec{x}(t_1) \longrightarrow 0$. This problem, or source of probable error, could be minimized by rescaling the amplifiers used to compute $\vec{y}(t)$ and $\vec{w}(t)$ when switching from forward time to reverse time, or by using separate amplifiers to compute the two functions.

The only analytic work performed for this plant was the computation of the time t_1 and the results are presented in Table V-V. As can be seen from the table, there is good agreement in all cases between the analytic and computer values of the time t_1.

Table V-V

Comparison of Analytic and Computer Values of the Time t_1

Initial state	analytic	computer
$\vec{x}(0) = \begin{Bmatrix} 0.2 \\ 0 \end{Bmatrix}$	0.64	0.6449
$\vec{x}(0) = \begin{Bmatrix} 3.5 \\ 0 \end{Bmatrix}$	1.55	1.549
$\vec{x}(0) = \begin{Bmatrix} 30 \\ 0 \end{Bmatrix}$	2.49	2.463

Plant 3)

The computer was set up as shown in Figure V-30 using the basic circuits of Section IV for

(V-21) $$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -.2 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u \ , \ |u| \leqslant 1$$

Initial states of $\vec{x}(0) = \begin{Bmatrix} 2 \\ 0 \end{Bmatrix}$, $\begin{Bmatrix} 7 \\ 0 \end{Bmatrix}$, and $\begin{Bmatrix} 6 \\ 6 \end{Bmatrix}$ were selected to provide systems requiring differing numbers of control signal switching. The criterion used for the selection of $\Delta\tau$ was $\Delta\vec{\eta}_{i+1} \cdot \Delta\vec{\eta}_i > 0$ for all three initial states. The data are presented in Table V-VI, the plots of η_{i2} versus η_{i1} are presented in Figures V-31, V-32,

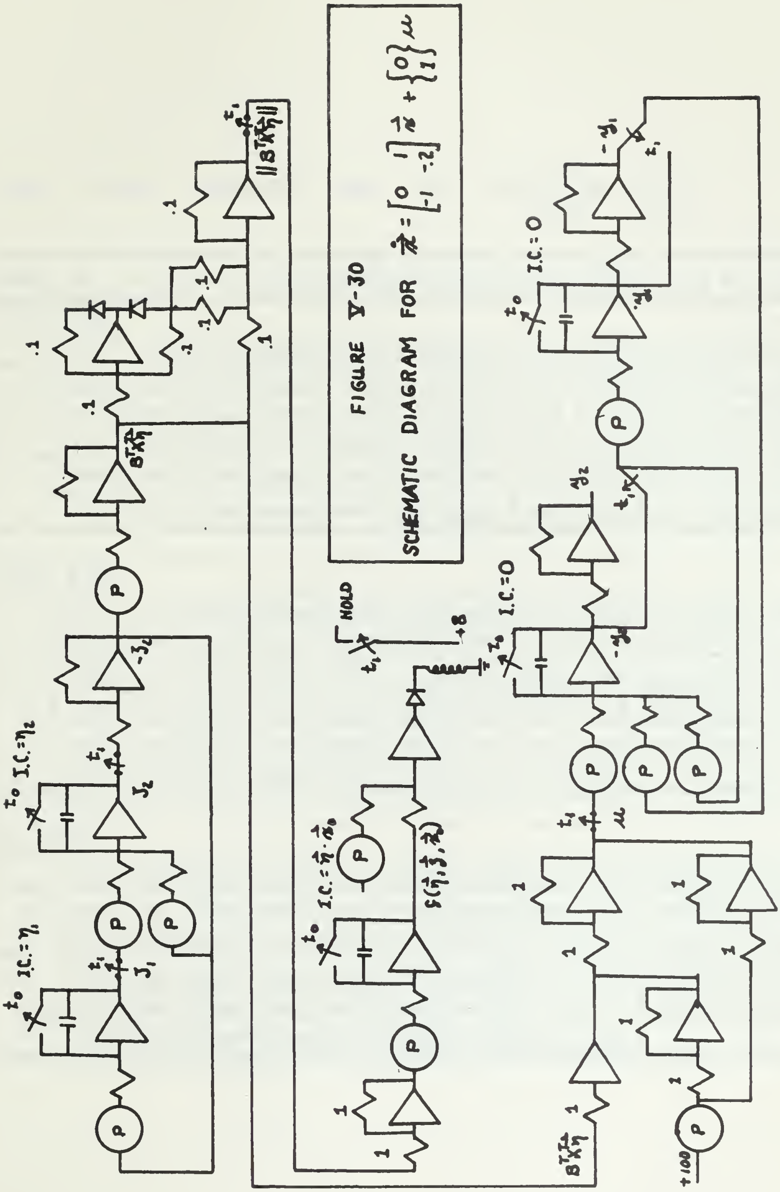


FIGURE V-30

SCHEMATIC DIAGRAM FOR $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -.2 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \mu$

TABLE V-VI COMPUTER DATA FOR $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -.2 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \mu$

RUN	η_1	η_2	t_1	$x_1(t_1)$	$x_2(t_1)$	$z_1(t_1)$	$z_2(t_1)$	$d\eta_1/dt$	$d\eta_2/dt$	Δt	$\Delta\eta_1$	$\Delta\eta_2$
$\vec{x}_0 = \begin{Bmatrix} 2 \\ 0 \end{Bmatrix}$												
1	-1.0	0	2.372	+.4878	-.5425	-1.983	+.899	-.017	-.899	0.25	-.004	-.224
2	-1.004	-.224	2.513	+.1806	-.1772	-2.051	+.329	+.051	-.329	0.25	+.013	-.082
3	-.991	-.306	2.534	+.0706	-.0595	-2.022	+.139	+.022	-.139	0.25	+.005	-.035
4	-.986	-.341	2.540	+.0291	-.0256	-2.002	+.065	+.002	-.065	0.25	+.001	-.016
5	-.985	-.357	2.539	+.0053	-.0065	-1.987	+.031	-.013	-.031	0.25	-.003	-.008
6	-.988	-.365	2.540	+.0055	+.0074	-1.986	+.022	-.014	-.022	0.25	-.003	-.005
$\vec{x}_0 = \begin{Bmatrix} 7 \\ 0 \end{Bmatrix}$												
1	-1.0	0	7.569	+.8720	+.2219	-6.871	+2.279	-.129	-2.279	0.125	-.016	-.284
2	-1.016	-.284	7.679	-.1432	-.0703	-6.861	+.019	-.139	-.019	0.125	-.017	-.002
3	-1.033	-.286	7.679	-.1221	-.0689	-6.860	+.090	-.140	-.090	0.125	-.017	-.011
4	-1.050	-.297	7.679	-.1473	-.0789	-6.859	+.063	-.141	-.063	0.125	-.017	-.008
5	-1.067	-.305	7.679	-.1499	-.0839	-6.839	+.001	-.161	-.001	0.125	-.020	0
6	-1.087	-.305	7.679	-.1389	-.0739	-6.851	-.030	-.149	+.030			
$\vec{x}_0 = \begin{Bmatrix} 6 \\ 6 \end{Bmatrix}$												
1	-.707	-.707	8.799	+.3049	-.0919	-6.699	-5.039	+.699	-.961	0.0625	+.044	-.060
2	-.663	-.767	8.669	-.0989	-.1879	-5.619	-5.669	-.381	-.331	.0625	-.024	-.021
3	-.687	-.788	8.799	+.0419	-.0539	-6.199	-5.519	+.199	-.481	.0625	+.013	-.030
4	-.676	-.818	8.729	-.0619	-.1269	-5.829	-5.669	-.171	-.331	.0625	-.011	-.021
5	-.687	-.839	8.809	-.0979	-.0439	-5.889	-5.839	-.111	-.161	.0625	-.007	-.010
6	-.693	-.848	8.809	-.0983	-.0429	-5.889	-5.849	-.101	-.151			

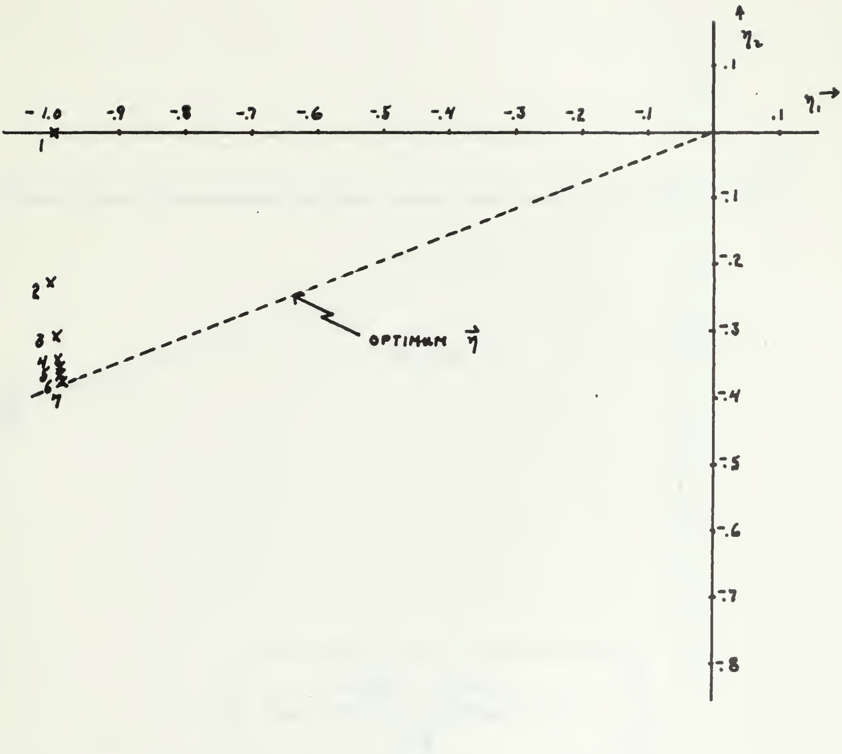


FIGURE V-31 PLOT OF η_{a2} VERSUS η_{a1} FOR

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -.2 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u, \quad \vec{x}_0 = \begin{Bmatrix} 2 \\ 0 \end{Bmatrix}$$

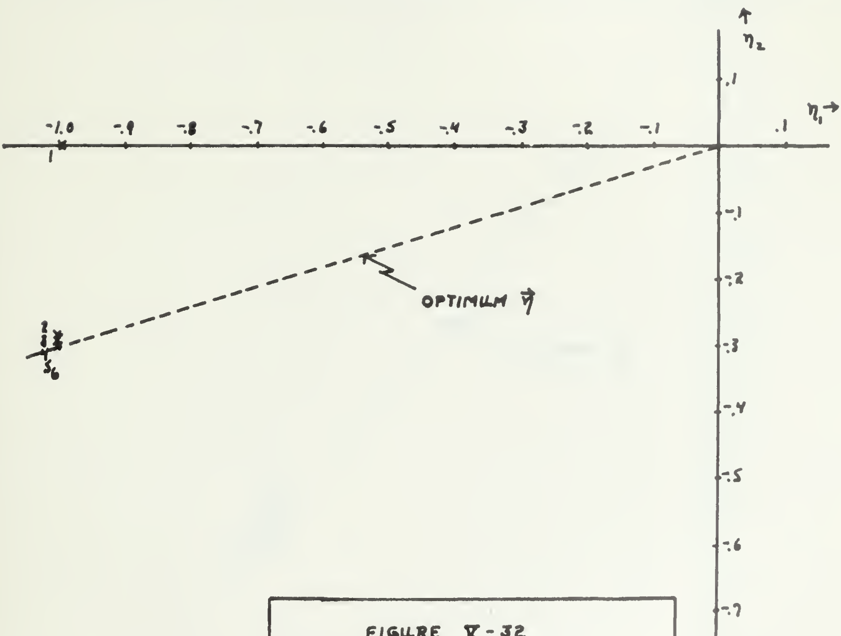


FIGURE V - 32
PLOT OF η_{a2} VERSUS η_{a1}
FOR

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -.2 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u \;, \vec{x}_0 = \begin{Bmatrix} 7 \\ 0 \end{Bmatrix}$$

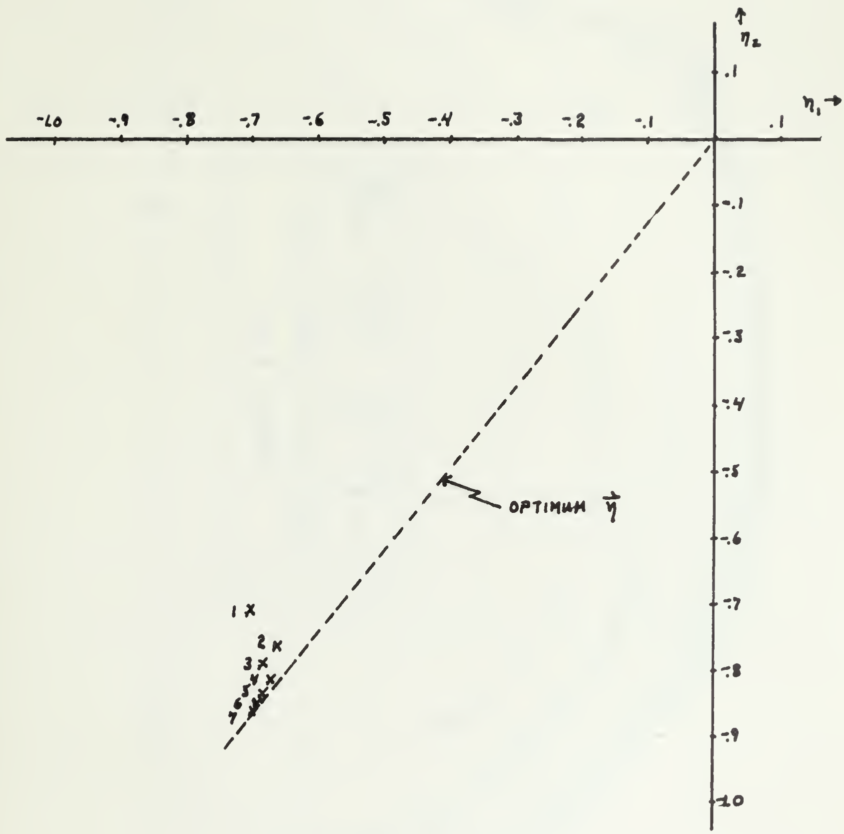


FIGURE V-33

PLOT OF η_{u2} VERSUS η_{u1} FOR

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -.2 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u \, , \quad \vec{x}_0 = \begin{Bmatrix} 6 \\ 6 \end{Bmatrix}$$

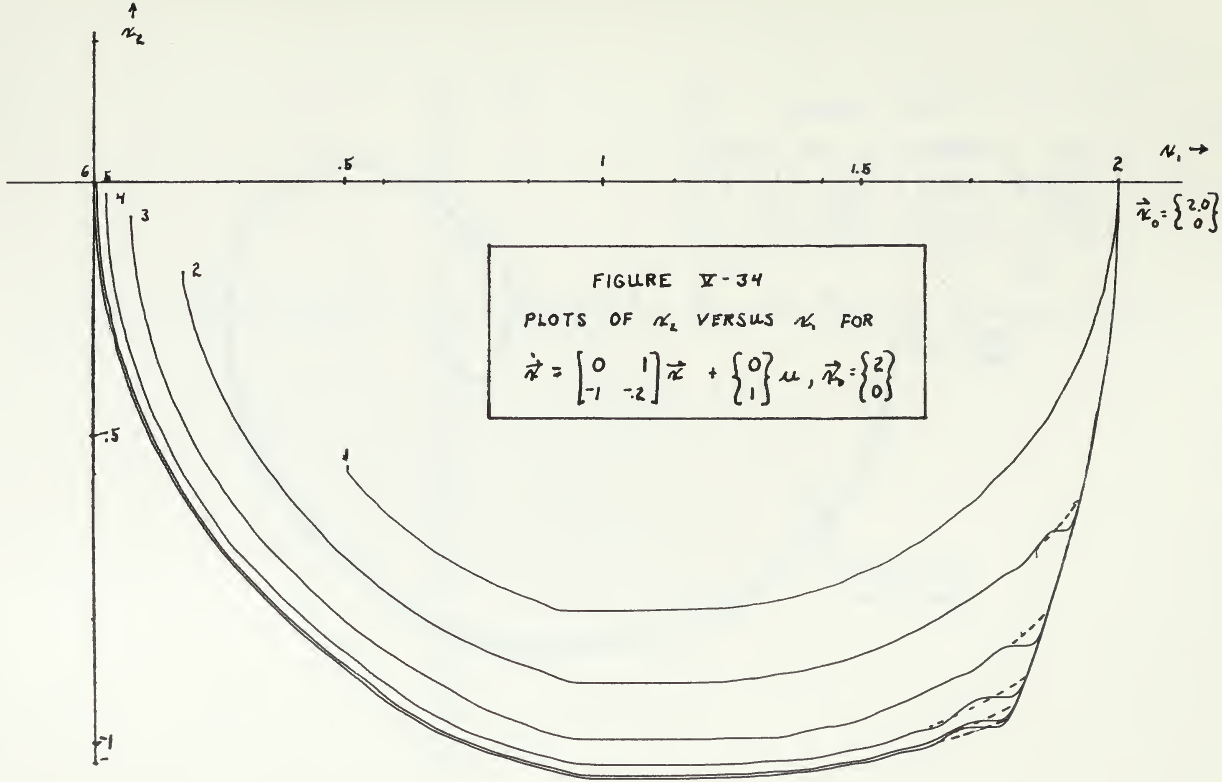


FIGURE V-34

PLOTS OF x_2 VERSUS x_1 FOR

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -.2 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u, \quad \vec{x}_0 = \begin{Bmatrix} 2 \\ 0 \end{Bmatrix}$$

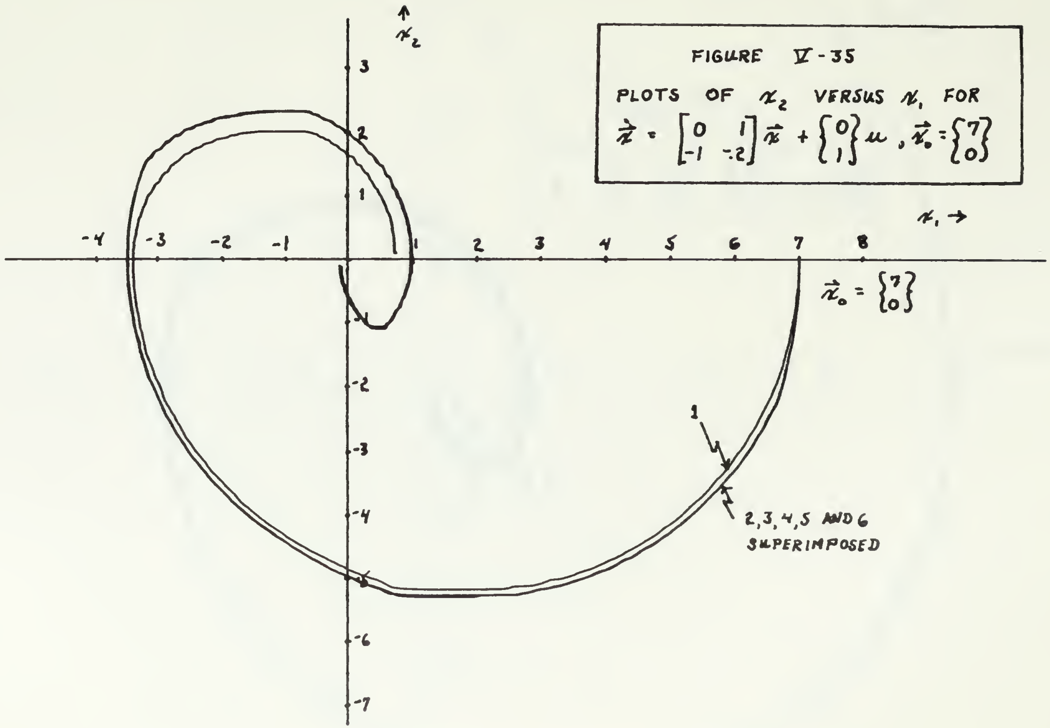


FIGURE V-35

PLOTS OF x_2 VERSUS x_1 FOR

$$\dot{\vec{x}} = \begin{bmatrix}0 & 1\\ -1 & -.2\end{bmatrix}\vec{x} + \begin{Bmatrix}0\\1\end{Bmatrix}u, \quad \vec{x}_0 = \begin{Bmatrix}7\\0\end{Bmatrix}$$

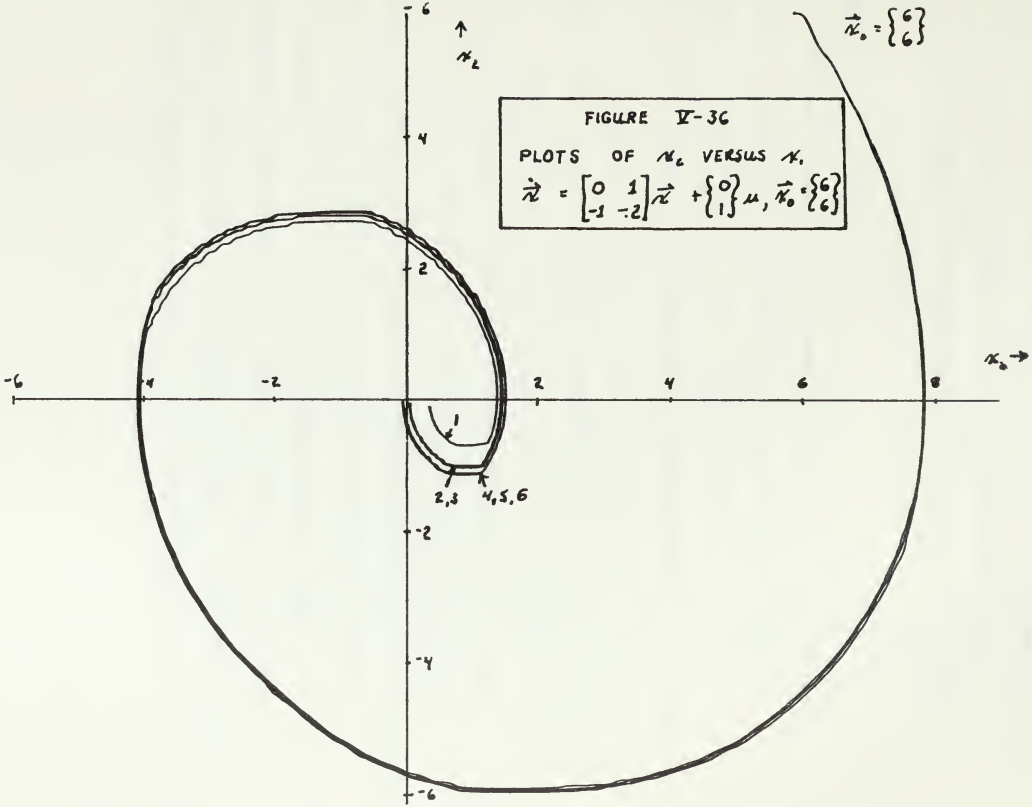


FIGURE V-36

PLOTS OF x_2 VERSUS x_1

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u, \quad \vec{x}_0 = \begin{Bmatrix} 6 \\ 6 \end{Bmatrix}$$

and V-33 respectively for the three given initial states and the plots of $x_2(t)$ versus $x_1(t)$ are presented in Figures V-34, V-35, and V-36. In all three cases the convergence to $\vec{\eta}^{\,o}$ was rapid and $\vec{x}(t_1)$ and $(\vec{z}(t_1, \vec{\eta}) + \vec{x}_0)$ were of the same order of magnitude as $\vec{\eta} \longrightarrow \vec{\eta}^{\,o}$.

Plant 4)

The computer was set up as shown in Figure V-37 using the basic circuits of Section IV modified for the three-dimension system

$$\text{(V-22)} \qquad \dot{\vec{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} u \ , \ |u| \leqslant 1$$

$\vec{x}(0) = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$ was the initial state which was investigated. Initially the criterion for the selection of $\Delta\tau$ was that $\Delta\vec{\eta}_{i+1} \cdot \Delta\vec{\eta}_i > 0$, however, after seven iterations it was decided to use as the criterion, $\Delta\tau_{max}$ which would yield $t_{1_{i+1}} > t_{1_i}$. Using this criterion, a total of sixty iterations were completed at which time $\vec{x}(t_1) \not\rightarrow 0$ and $\vec{z}(t_1) \not\rightarrow -\vec{x}(0)$. The data for iterations fifty through sixty are presented in Table V-VII. There was no apparent pattern for the selection of $\Delta\tau$. In all iterations it was possible to duplicate values of t_1 for successive re-iterations with the same computer parameters; it was not possible to duplicate values of $\vec{z}(t_1)$. With this in mind some error analysis work is indicated to détermine the effect of an error in $\vec{z}(t_1)$, introduced by some random signal, has on the convergence to $\vec{\eta}^{\,o}$.

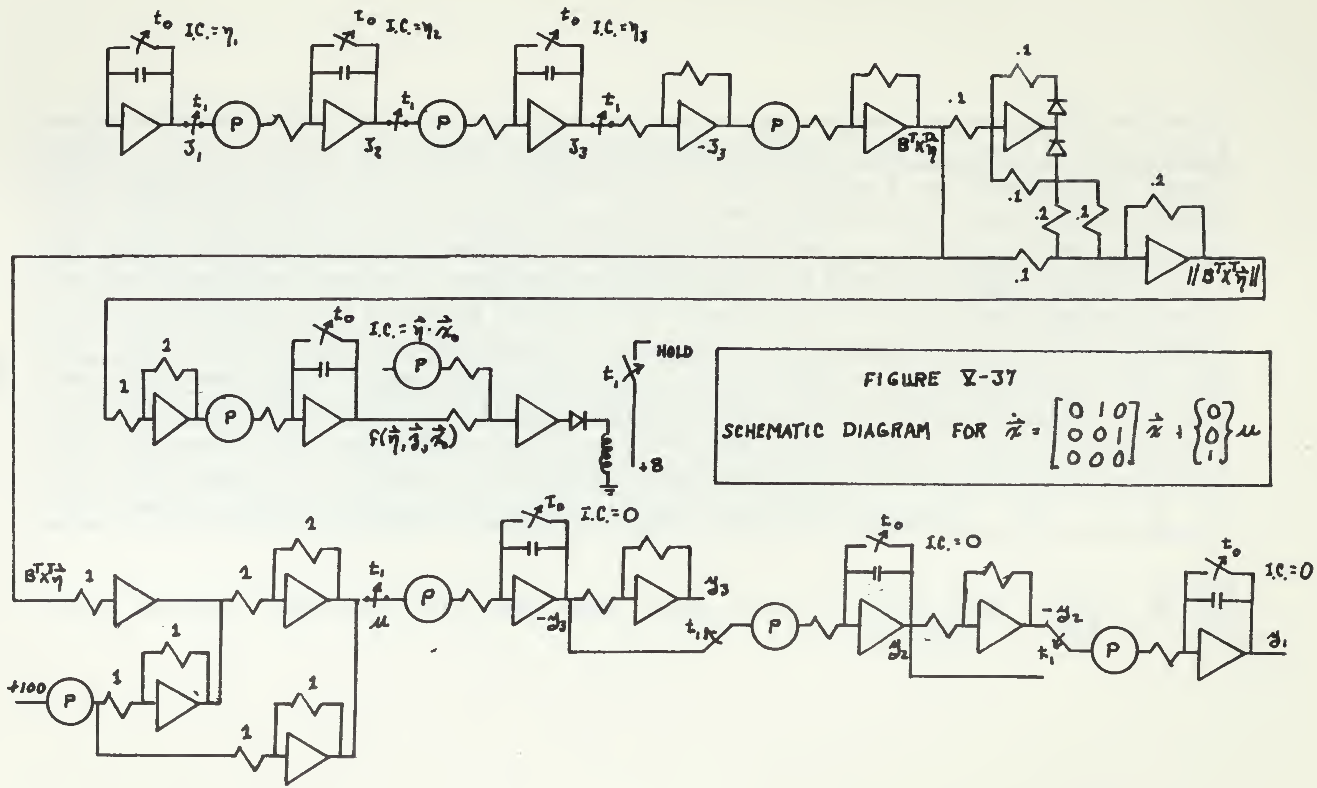


FIGURE V-37

SCHEMATIC DIAGRAM FOR $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} u$

RUN	η_1	η_2	η_3	t_1	$x_1(t_1)$	$x_2(t_1)$	$x_3(t_1)$	$\lambda_1(t_1)$	$\lambda_2(t_1)$	$\lambda_3(t_1)$	$d\eta_1/dt$	$d\eta_2/dt$	$d\eta_3/dt$	Δt	$\Delta\eta_1$	$\Delta\eta_2$	$\Delta\eta_3$
50	-1.004	-4.169	-7.334	7.101	+11.79	+17.21	+4.847	-2.707	-.899	-.775	+1.707	-.101	-.225	1/256	+.007	.000	-.001
51	-.997	-4.169	-7.335	7.103	+13.04	+21.90	+9.745	+.658	-2.130	-.539	-1.658	+1.130	-.461	1/256	-.006	+.004	-.002
52	-1.003	-4.165	-7.337	7.104	+11.14	+16.35	+4.773	-2.502	-.989	-.767	+1.502	-.011	-.223	1/512	+.006	.000	-.001
53	-.997	-4.165	-7.338	7.106	+11.65	+18.62	+7.359	-.477	-1.721	-.632	-.523	+.721	-.367	1/128	-.004	+.006	-.003
54	-1.001	-4.159	-7.341	7.109	+11.20	+16.89	+5.647	-1.737	-1.254	-.730	+.737	+.254	-.270	1/256	+.003	+.001	-.001
55	-.998	-4.158	-7.342	7.112	+11.33	+17.51	+.6464	-1.127	-1.485	-.690	+.127	+.485	-.310	1/4	+.032	+.096	-.080
56	-.966	-4.062	-7.422	7.121	+5.045	-1.652	-3.570	-5.775	+.536	-1.191	+4.775	-.462	+.191	1/512	+.010	-.001	.000
57	-.956	-4.063	-7.422	7.129	+6.655	+7.447	+2.070	-2.003	-.8507	-.922	+1.003	-.149	-.078	1/256	+.004	.000	.000
58	-.952	-4.063	-7.422	7.130	+7.102	+9.311	+4.147	-.341	-1.445	-.809	-.659	+.445	-.191	1/128	-.005	+.004	-.002
59	-.957	-4.059	-7.424	7.133	+.6354	+6.311	+.882	-3.037	-.473	-.998	+2.037	-.527	-.002	1/256	+.008	-.002	.000
60	-.949	-4.061	-7.424	7.135	+7.383	+10.14	+5.143	+.311	-1.680	-.766	-1.311	+.680	-.234				

TABLE V-VII COMPUTER DATA FOR RUNS 50-60 FOR $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} + \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} u \,,\ \vec{x}_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$

VI. SUMMARY OF RESEARCH RESULTS

The optimum control law for two-dimension systems is readily obtained using Neustadt's synthesis technique as adapted to analog computer solution. For the solution of three-dimension and higher order systems some refinement of the computer technique used in this research is necessary.

The initial choice of $\vec{\eta}$ has relatively little effect on the convergence to the optimum steering order. There are a wide variety of criteria for the selection of $\Delta\tau$ which in general lead to satisfactory results for the two-dimension system. For design of an automatic system perhaps the criterion that $\Delta\vec{\eta}_{i+1} \cdot \Delta\vec{\eta}_i > 0$ would be the best suited. In general large values of $\Delta\tau$ give slow convergence with large oscillations about $\vec{\eta}^o$ while small values of $\Delta\tau$ give slow positive convergence. There obviously are some $\Delta\tau$ which give the best convergence but no optimization was attempted in this research work.

Using the finite difference equation in computing iterative values of $\vec{\eta}$ it becomes possible for $\vec{\eta}$ to leave the domain.

VII. POSSIBLE EXTENSIONS OF THIS RESEARCH WORK

1. The natural extension of this work is to continue on into the third and higher dimension systems.

2. A significant extension of this work would be the optimization of $\Delta\tau$.

3. An error analysis of the effect random errors in the computation of $\vec{z}(t_1)$ has on the convergence to $\vec{\eta}^{\,o}$ would be noteworthy.

4. To be of maximum use the system must be fully automatic, utilizing logic circuits, and/or a digital-analog combination which subject requires some investigation.